Context for models of concurrency

Peter Bubenik

peter.bubenik@epfl.ch

http://igat.epfl.ch/bubenik/talks/

Ecole Polytechnique Fédérale de Lausanne
(Swiss Federal Institute of Technology at Lausanne

Motivation

Concurrency

We would like to understand systems in which processes run concurrently.

Motivation

Concurrency

We would like to understand systems in which processes run concurrently.

In concurrent systems the use of shared resources may be constrained.

A concurrent system

Example:
2 processes using 2 shared resources a and b which can only be used by one process at a time

A concurrent system

Example:
2 processes using 2 shared resources a and b which can only be used by one process at a time

Notation:
Px - a process locks resource x
Vx - a process releases resource x

A concurrent system

Example:
2 processes using 2 shared resources a and b which can only be used by one process at a time

Notation:
Px - a process locks resource x
Vx - a process releases resource x

Program of the first process: $Pa \quad Pb \quad Vb \quad Va$

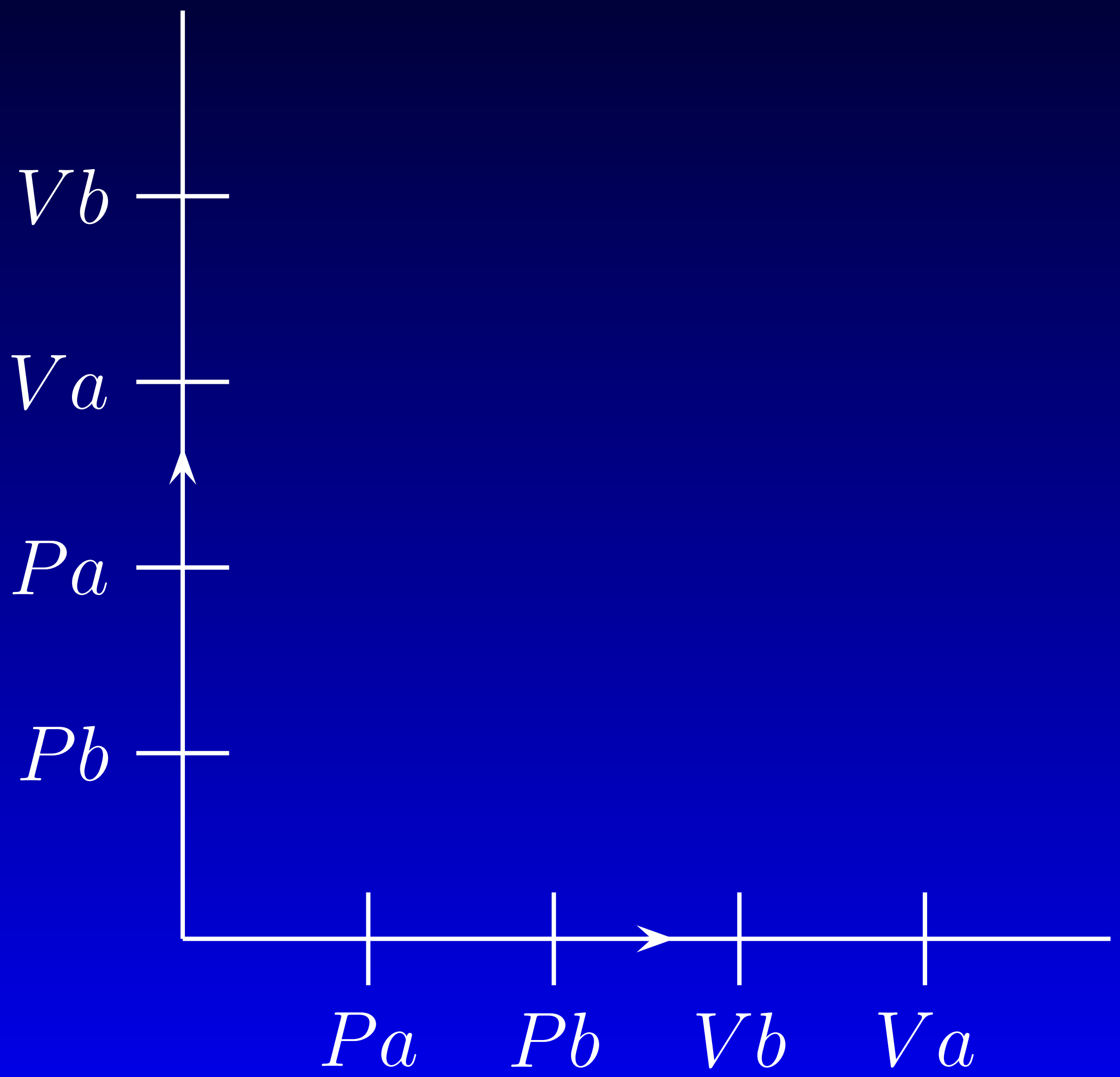
Program of the second process: $Pb \quad Pa \quad Va \quad Vb$

The model

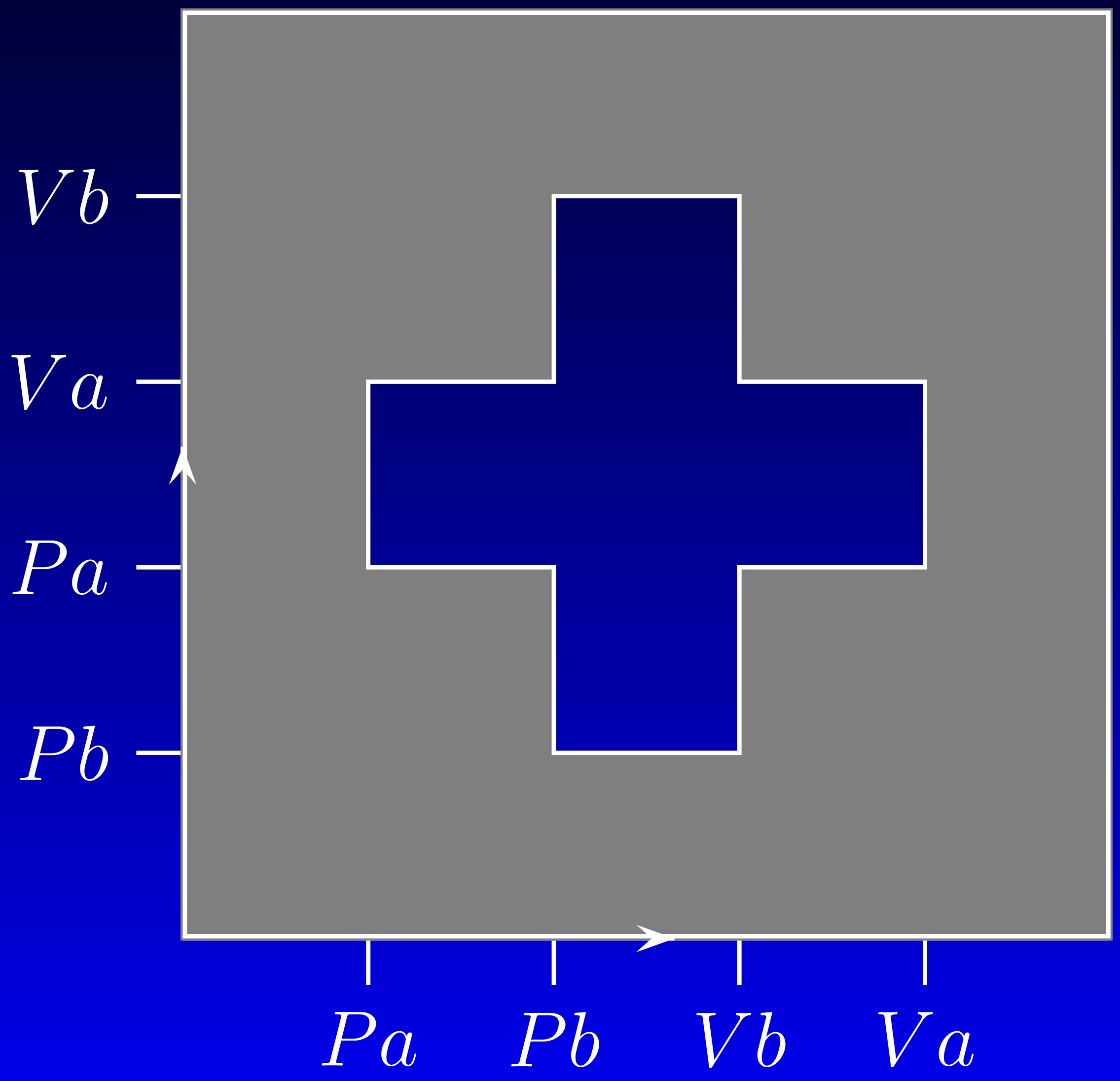
Concurrent systems can be modeled by a po-spaces.

Definition: A po-space is a topological space U with a partial order $\leq$ which is a closed subset of $U \times U$.

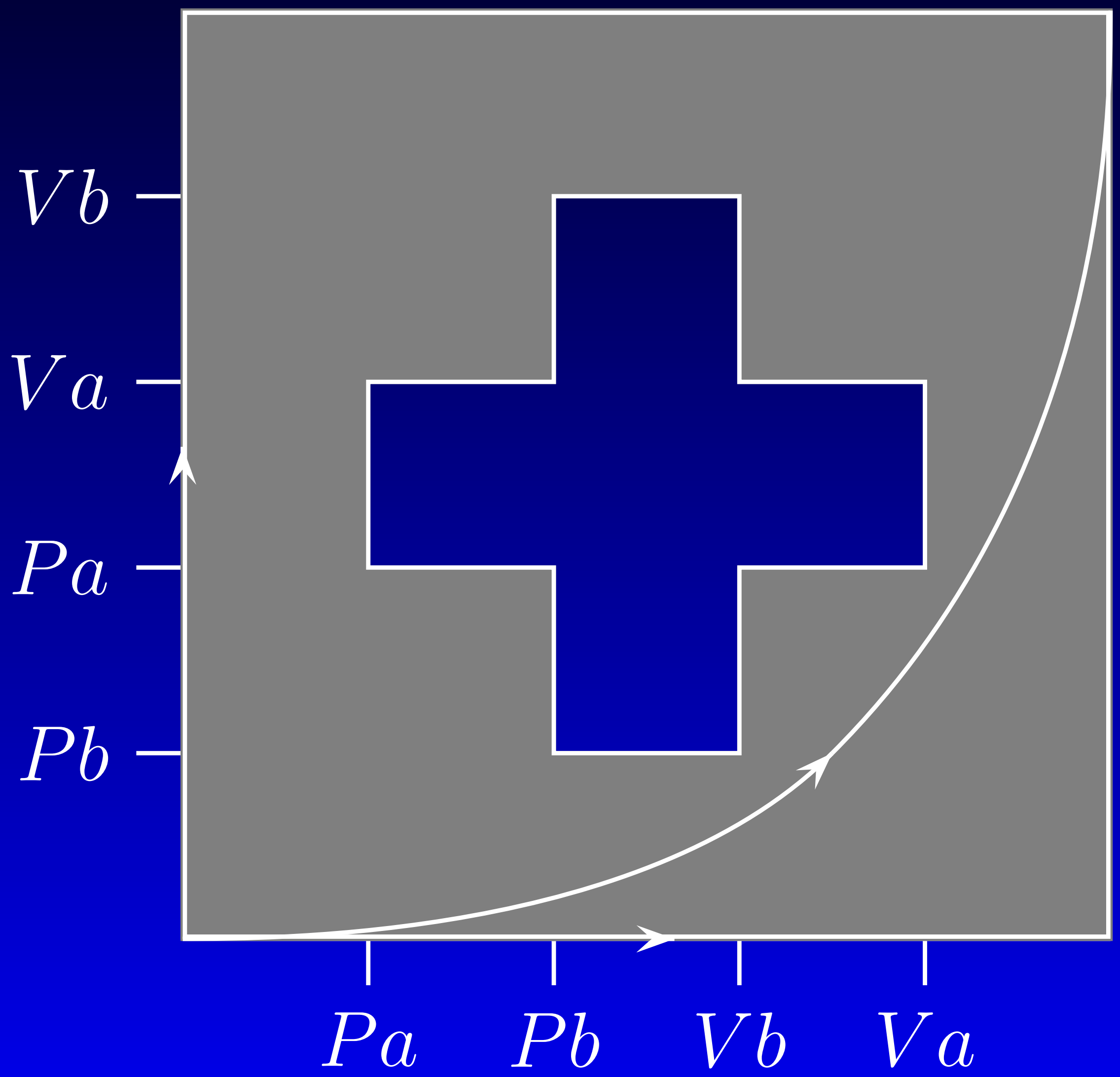
The swiss flag



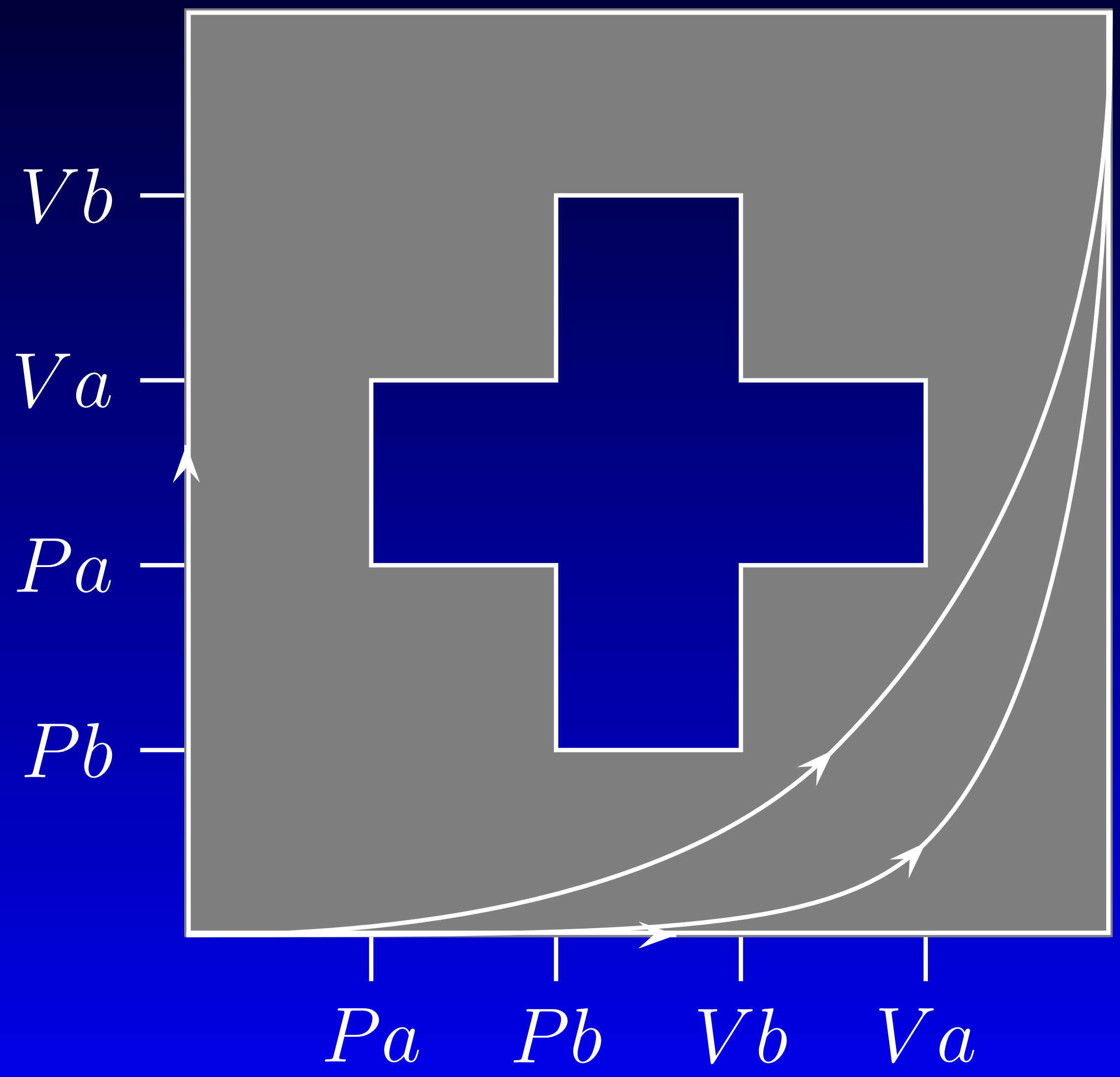
The swiss flag



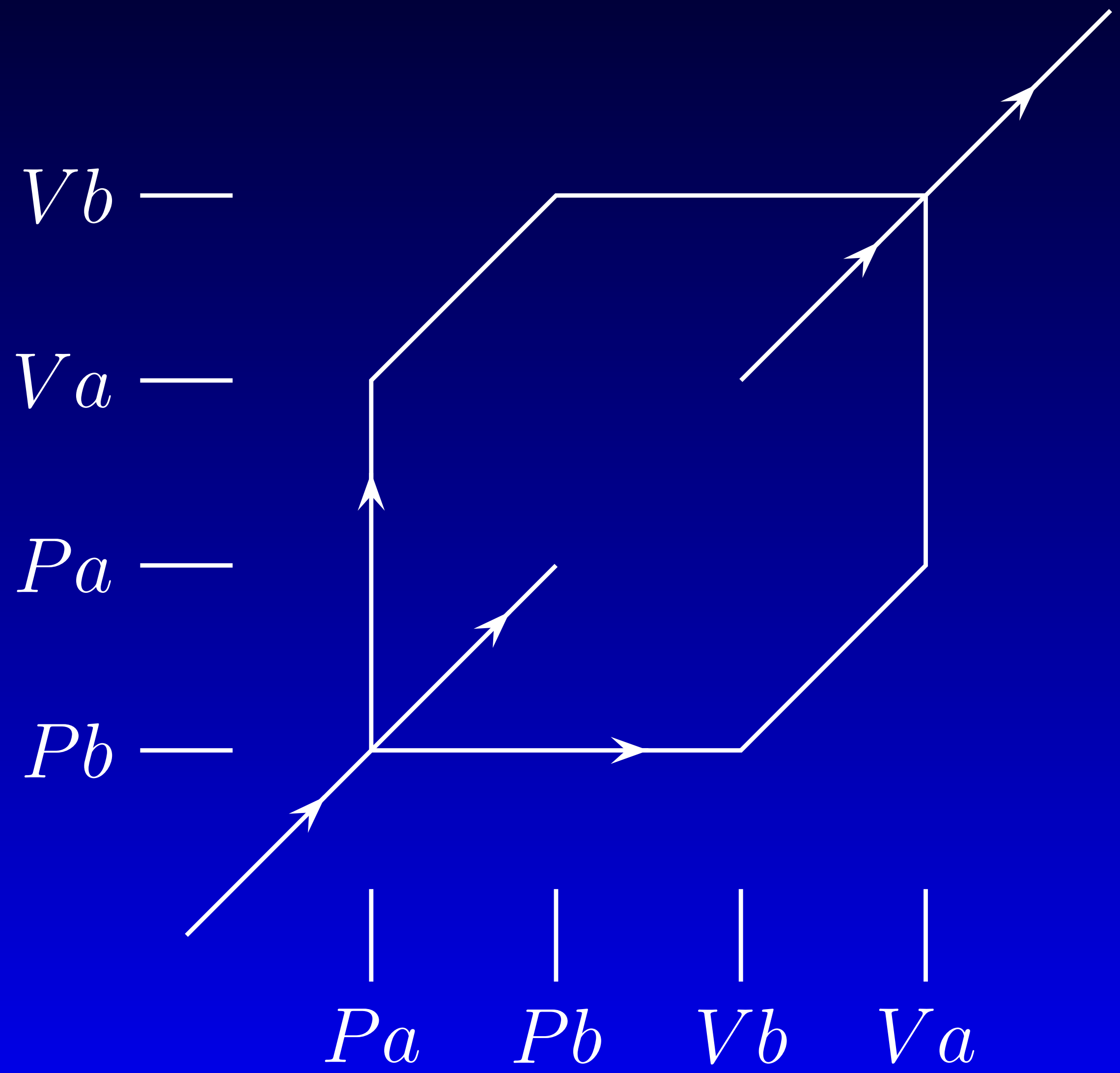
The swiss flag



The swiss flag



A sub-po-space of the swiss flag



Goal

Develop a framework for concurrency where equivalences are accounted for.

Goal

Develop a framework for concurrency where equivalences are accounted for.

We would like equivalences that allow a piece-by-piece analysis of po-spaces.

Goal

Develop a framework for concurrency where equivalences are accounted for.

We would like equivalences that allow a piece-by-piece analysis of po-spaces.

One approach: Use dihomotopy equivalences.

The category Pospace

Definition:

- A po-space is a topological space U with a partial order $\leq$ which is a closed subset of $U \times U$.

The category Pospace

Definition:

- A po-space is a topological space U with a partial order $\leq$ which is a closed subset of $U \times U$.
- A dimap is a continuous map $f : U_1 \to U_2$ between po-spaces such that

$$x \leq y \implies f(x) \leq f(y).$$

The category Pospace

Definition:

- A po-space is a topological space U with a partial order $\leq$ which is a closed subset of $U \times U$.
- A dimap is a continuous map $f : U_1 \to U_2$ between po-spaces such that

$$x \leq y \implies f(x) \leq f(y).$$

- Let **Pospace** be the category whose objects are po-spaces and morphisms are dimaps.

The category **Pospace**

Definition:

- A po-space is a topological space U with a partial order $\leq$ which is a closed subset of $U \times U$.
- A dimap is a continuous map $f : U_1 \to U_2$ between po-spaces such that

$$x \leq y \implies f(x) \leq f(y).$$

- Let **Pospace** be the category whose objects are po-spaces and morphisms are dimaps.

Remark: Subspaces and products of po-spaces inherit a po-space structure.

Dihomotopy equivalences

Definition:

- A dihomotopy between dimaps $f, g : B \to C$ is a dimap $\phi : B \times \vec{I} \to C$ such that $\phi|_{B\times\{0\}} = f$ and $\phi|_{B\times\{1\}} = g$. Write $\phi : f \to g$.

Dihomotopy equivalences

Definition:

- A dihomotopy between dimaps $f, g : B \longrightarrow C$ is a dimap $\phi : B \times \vec{I} \longrightarrow C$ such that $\phi|_{B\times\{0\}} = f$ and $\phi|_{B\times\{1\}} = g$. Write $\phi : f \longrightarrow g$.
- Write $f \simeq g$ if there is a chain of dihomotopies

$$f \longrightarrow f_1 \longleftarrow f_2 \longrightarrow \ldots \longleftarrow f_n \longrightarrow g.$$

Dihomotopy equivalences

Definition:

- A dihomotopy between dimaps $f, g : B \to C$ is a dimap $\phi : B \times \vec{I} \to C$ such that $\phi|_{B\times\{0\}} = f$ and $\phi|_{B\times\{1\}} = g$. Write $\phi : f \to g$.
- Write $f \simeq g$ if there is a chain of dihomotopies

$$f \to f_1 \leftarrow f_2 \to \dots \leftarrow f_n \to g.$$

- Po-Spaces B, C are dihomotopy equivalent if there are dimaps $f : B \leftrightarrows C : g$ such that

$$g \circ f \simeq \mathrm{Id}_B \text{ and } f \circ g \simeq \mathrm{Id}_C \,.$$

Using dihomotopy equivalences

Recall: We wanted to use dihomotopy equivalences to provide equivalences of concurrent systems.

Using dihomotopy equivalences

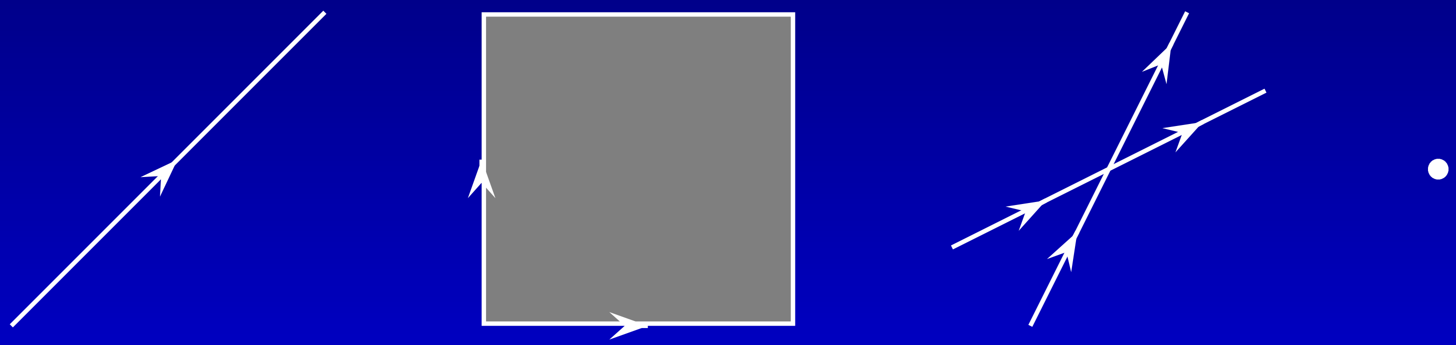
Recall: We wanted to use dihomotopy equivalences to provide equivalences of concurrent systems.

However all of the following spaces are dihomotopy equivalent.

Using dihomotopy equivalences

Recall: We wanted to use dihomotopy equivalences to provide equivalences of concurrent systems.

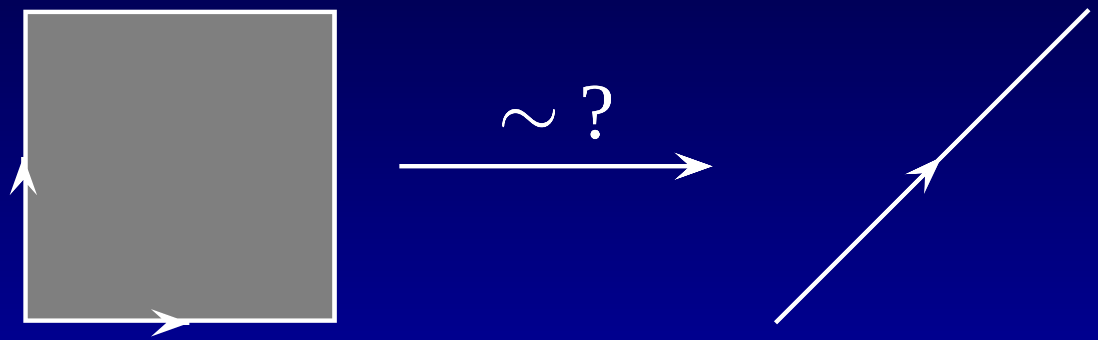
However all of the following spaces are dihomotopy equivalent.



Thus a stronger notion of equivalence is needed.

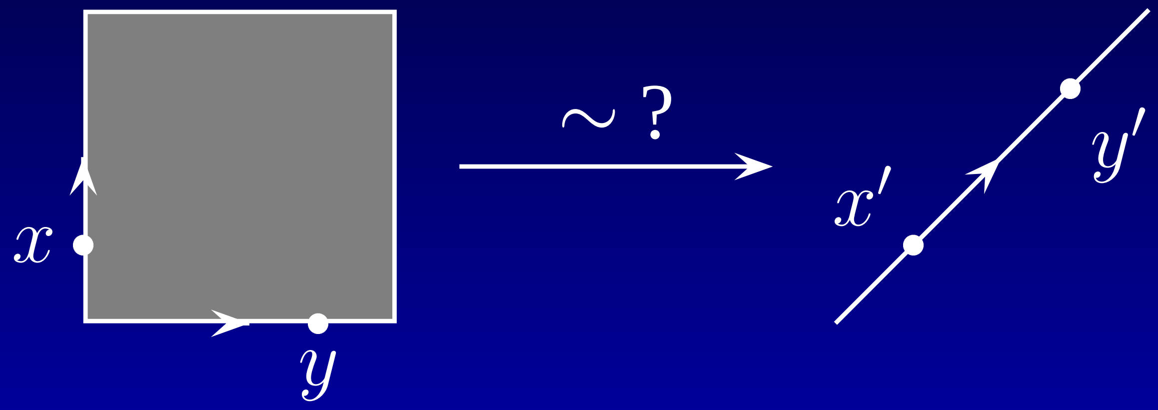
A basic example

Question: Is there an equivalence between $\vec{I} \times \vec{I}$ and $\vec{I}$?



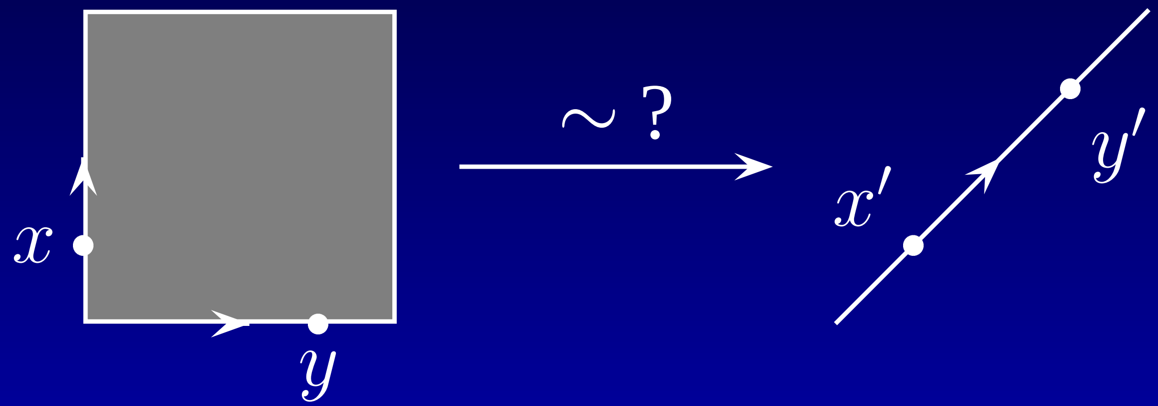
A basic example

Question: Is there an equivalence between $\vec{I} \times \vec{I}$ and $\vec{I}$?



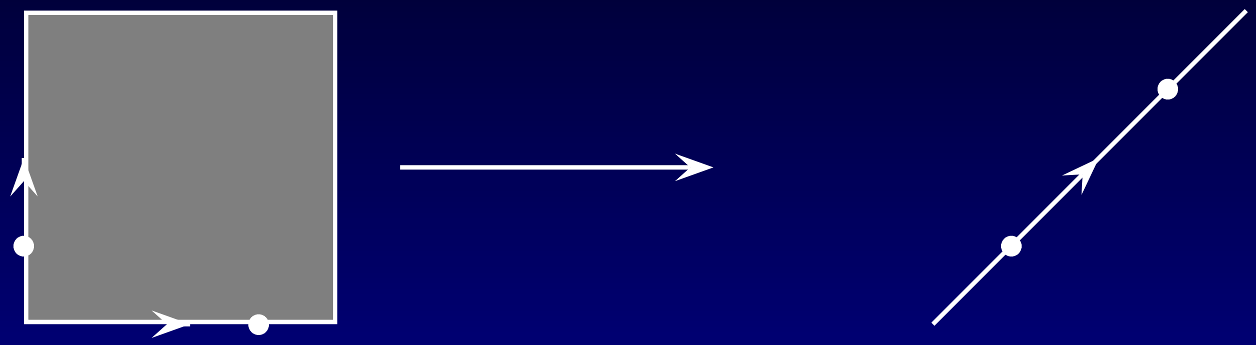
A basic example

Question: Is there an equivalence between $\vec{I} \times \vec{I}$ and $\vec{I}$?

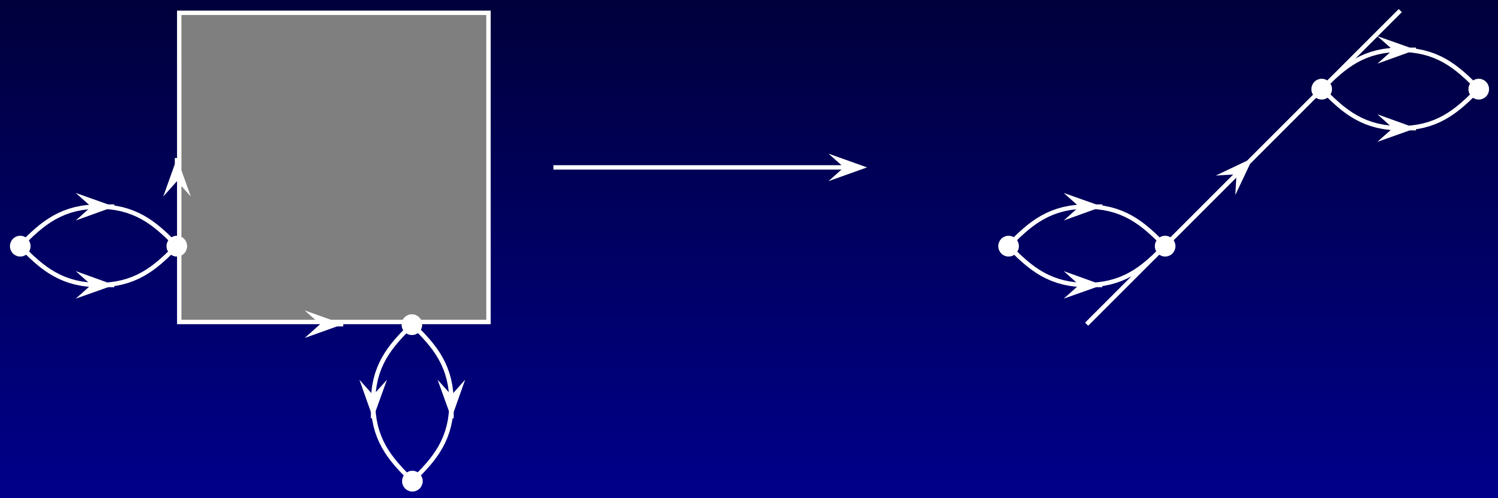


Our guide: If this map is an equivalence, then so is the map obtained by "pasting" po-spaces to this example. (Formally, we want the set of equivalences to be closed under pushouts with inclusions.)

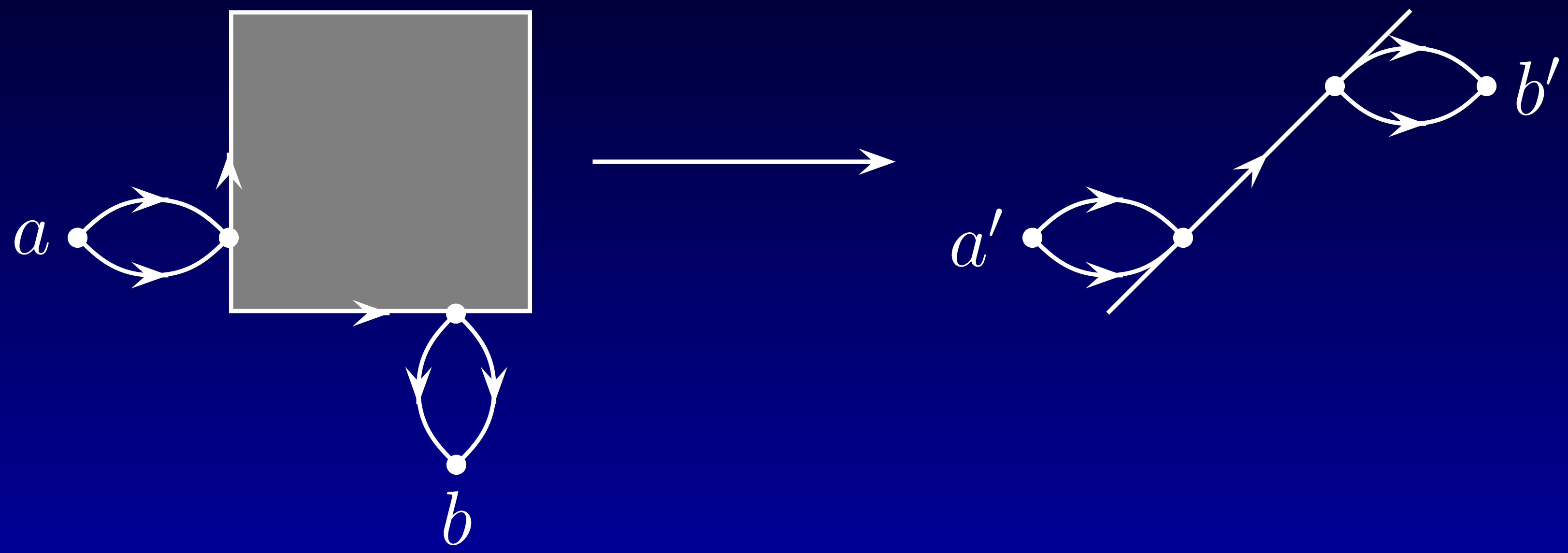
Adding to the basic example



Adding to the basic example



Adding to the basic example



There is no execution path from a to b, while there are such paths from a' to b'.
Thus this map should not be an equivalence.

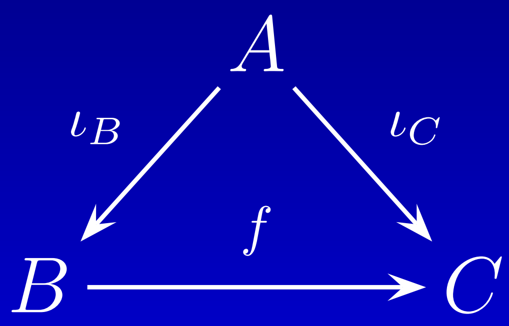
Possible Solution

Use context. That is, instead of working in the category **Pospace** work with the following category.

Possible Solution

Use context. That is, instead of working in the category **Pospace** work with the following category.

Definition: Given a po-space A (called the context), let $\mathbf{A} \downarrow \mathbf{Pospace}$ be the category whose objects are dimaps $\iota_B : A \to B \in \mathbf{Pospace}$ and whose morphisms are dimaps



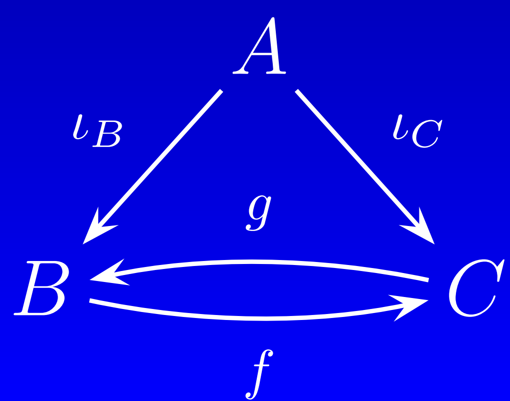
such that $f \circ \iota_B = \iota_C$

Remark: The choice of A depends on the pastings that one wants to consider.

Equivalences in $\mathbf{A} \downarrow$ **Pospace**

Definition:

- A dihomotopy between $f, g : B \to C$ $\in \mathbf{A} \downarrow$ **Pospace** is a dihomotopy $\phi : f \to g \in$ **Pospace** such that $\forall a \in A$, $\phi(\iota_B(a), t) = \iota_C(a)$.
- Write $f \simeq g$ if there is a chain of dihomotopies $f \to f_1 \leftarrow f_2 \to \ldots \leftarrow f_n \to g$.
- $\iota_B : A \to B, \iota_C : A \to C$ are dihomotopy equivalent if there are dimaps



such that $g \circ f \simeq \mathrm{Id}_B$

and $f \circ g \simeq \mathrm{Id}_C$.

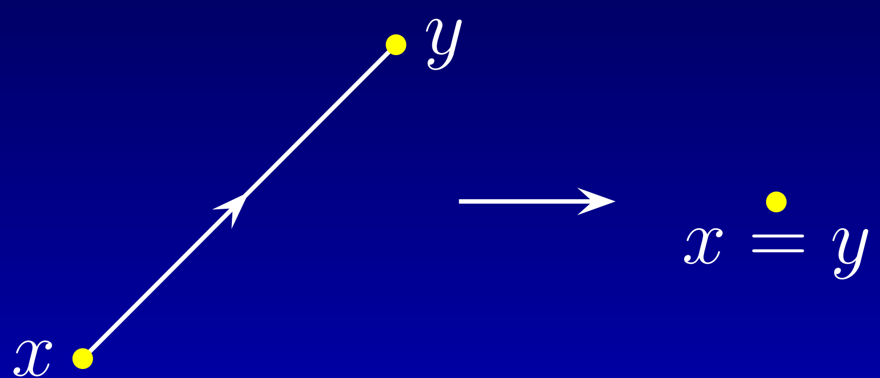
Example

$A = \{x, y\}$.
Then $\mathbf{A} \downarrow \mathbf{Pospace}$ is the category of po-spaces with two marked points.

Example

$A = \{x, y\}$.
Then $\mathbf{A} \downarrow \mathbf{Pospace}$ is the category of po-spaces with two marked points.

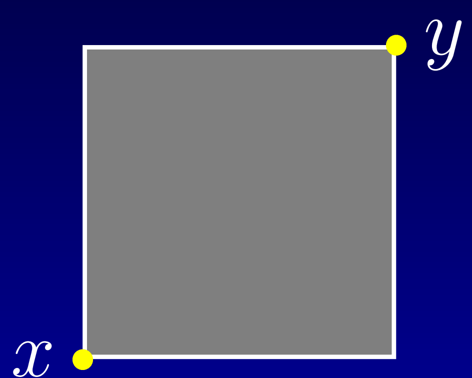
In this category



is not a dihomotopy equivalence since there is no dimap in the reverse direction.

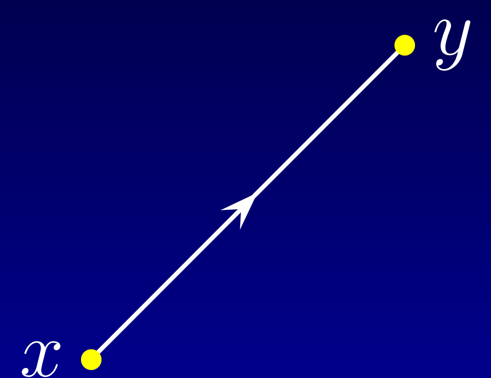
Example of an equivalence

In the same context (of two marked points) the dimaps



$$(t,t) \leftarrow t$$

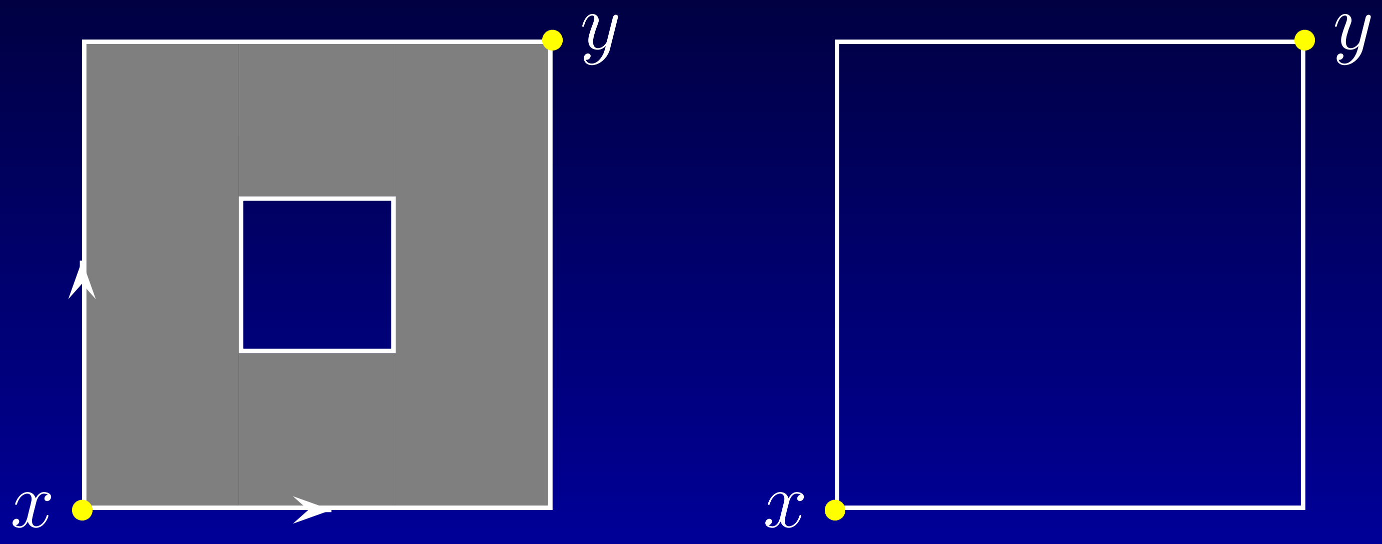
$$(x,y) \mapsto \max(x,y)$$



give a dihomotopy equivalence.

$\vec{I} \times \vec{I}$ with a square removed

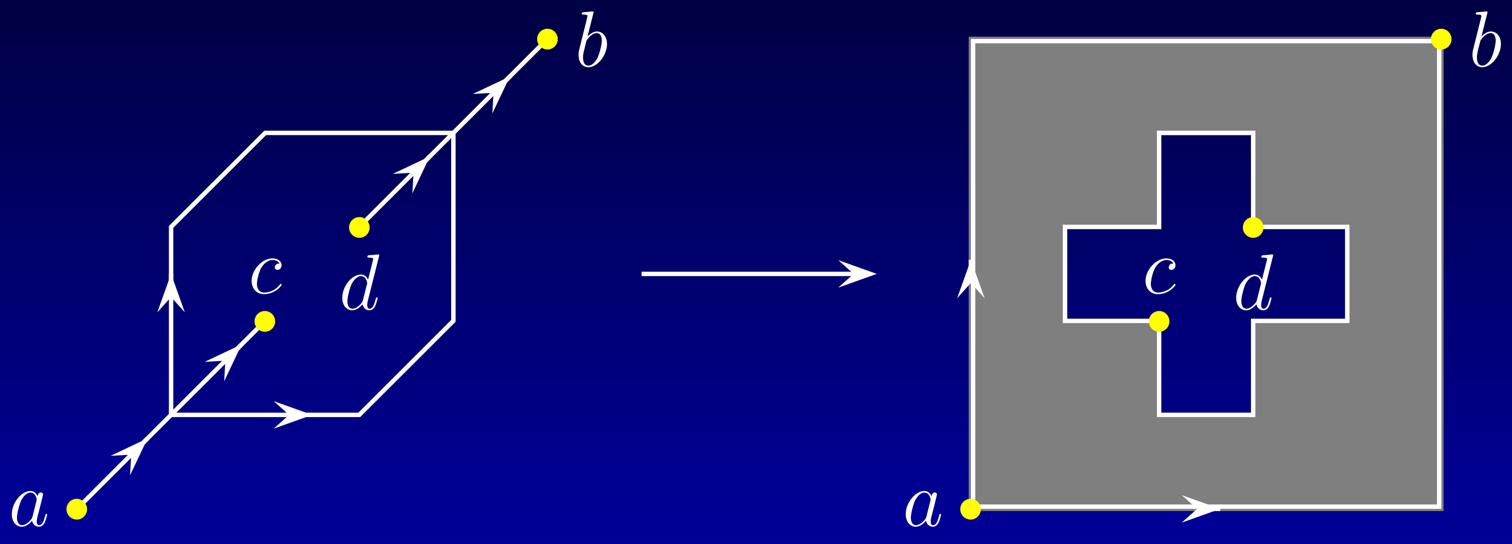
and two marked points



is dihomotopy equivalent to its boundary.

Context for the Swiss flag

Let $A = \{a, b, c, d\}$. Then the inclusion



is a dihomotopy equivalence in $\mathbf{A} \downarrow \mathbf{Pospace}$.

Dipaths

Definition: Let $x, y \in$ the po-space B.

- A dipath is a dimap $\vec{I} \longrightarrow B$.
- Dipaths are dihomotopy equivalent if they are so in the context of their endpoints.
- Let $\vec{\pi}_1(B)(x, y)$ be the set of dihomotopy equivalence classes of dipaths from x to y.

Dipaths in equivalent spaces

Notation: Given $f : B \to C \in \mathbf{A} \downarrow \mathbf{Pospace}$ and $x \in A$, let $x_B = \iota_B(x)$ and let $x_C = \iota_C(x)$.

Dipaths in equivalent spaces

Notation: Given $f : B \to C \in \mathbf{A} \downarrow \mathbf{Pospace}$ and $x \in A$, let $x_B = \iota_B(x)$ and let $x_C = \iota_C(x)$.

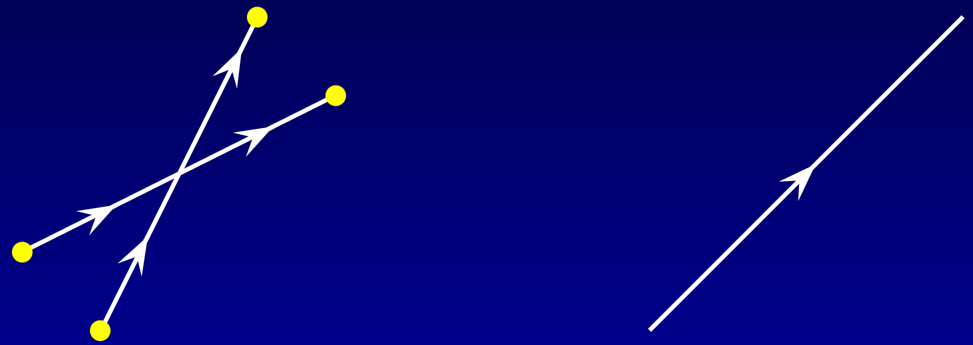
Proposition: Given $f : B \to C \in \mathbf{A} \downarrow \mathbf{Pospace}$ and $x, y \in A$ there is an induced map

$$\vec{\pi}_1(f)(x, y) : \vec{\pi}_1(B)(x_B, y_B) \to \vec{\pi}_1(C)(x_C, y_C).$$

If f is a dihomotopy equivalence then it is an isomorphism.

Example

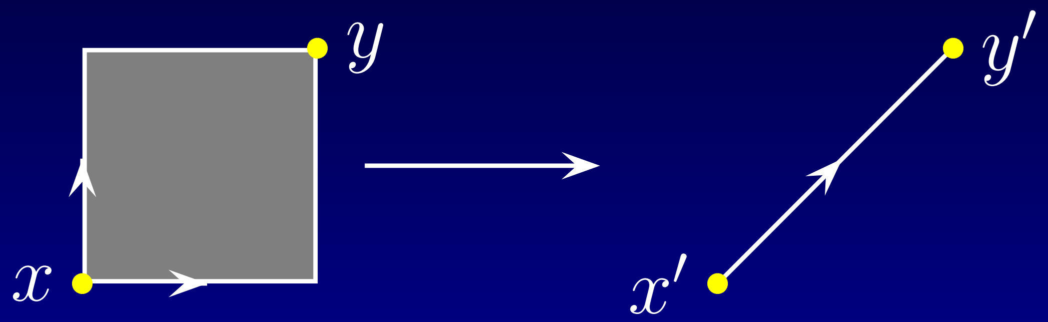
In the context of its four endpoints



the left hand po-space is not dihomotopy equivalent to $\vec{I}$.

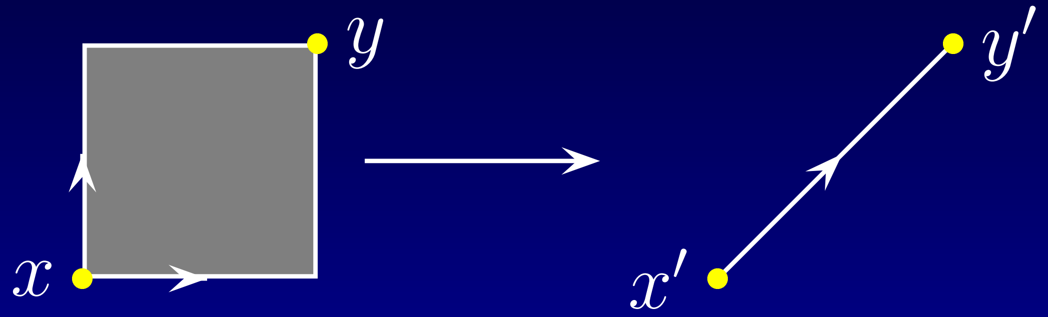
Another example

Recall that there is a dihomotopy equivalence

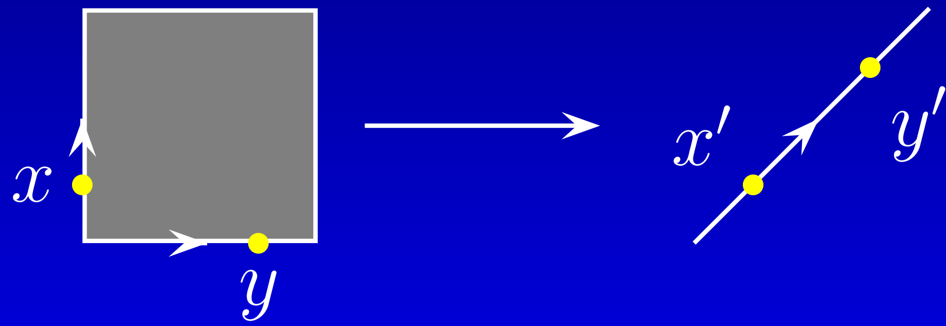


Another example

Recall that there is a dihomotopy equivalence

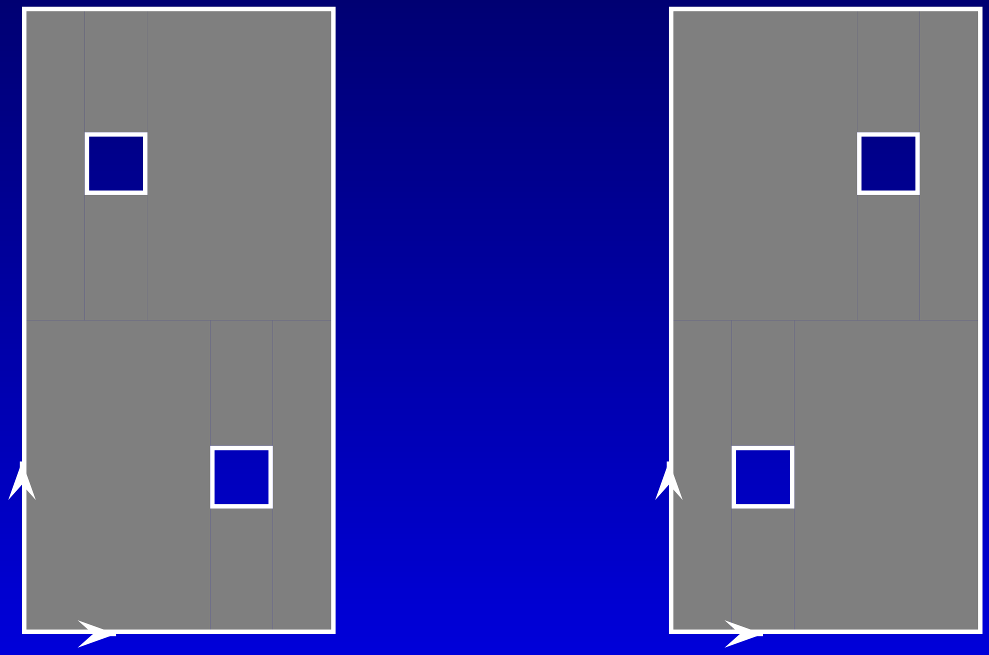


However there is no dihomotopy equivalence

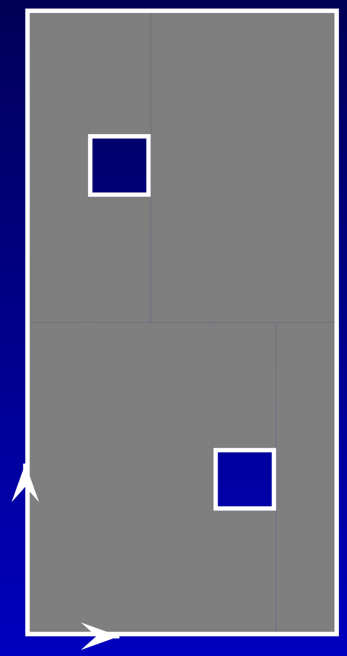


Compound examples

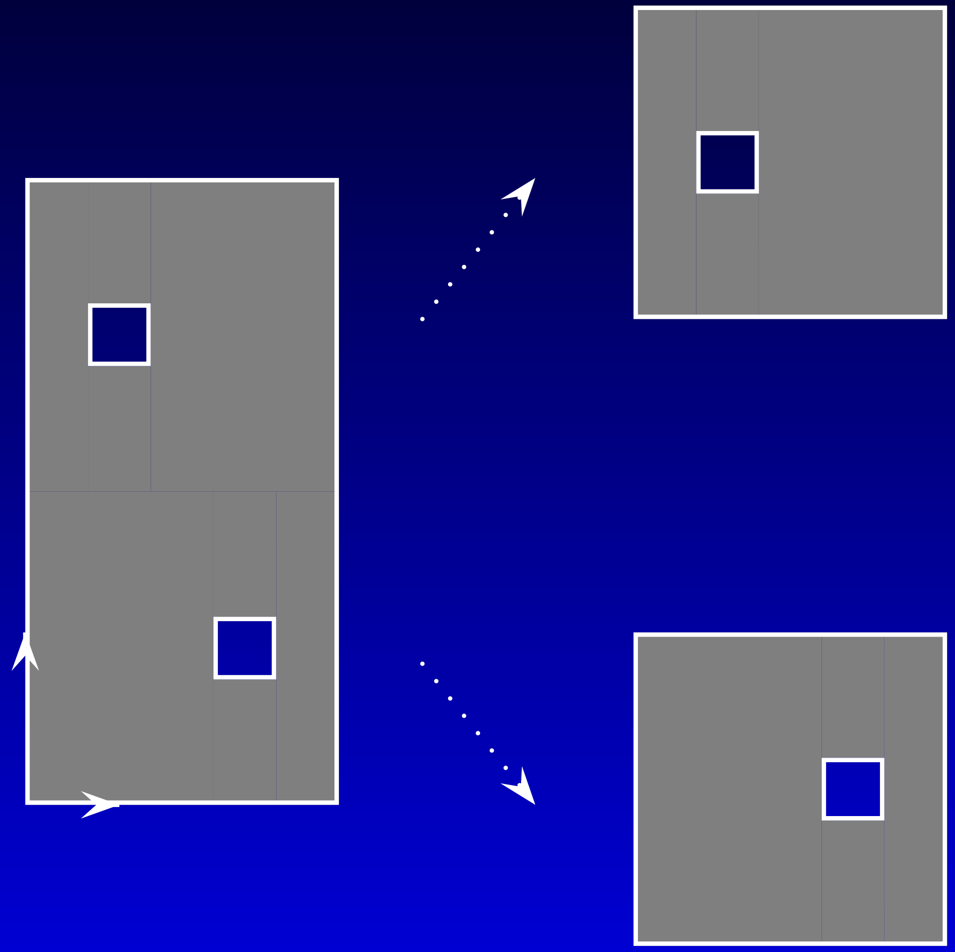
We would like to find equivalent po-spaces to the following examples by analyzing them piece-by-piece.



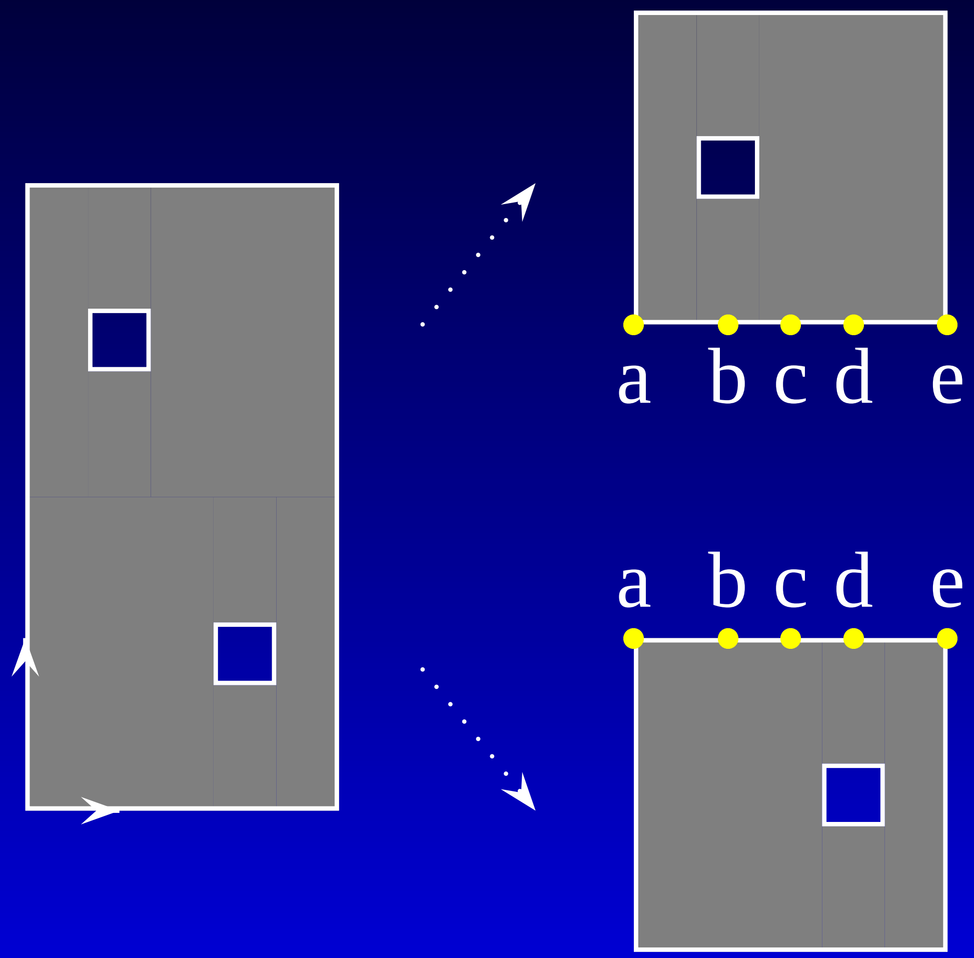
Equivalences of pieces



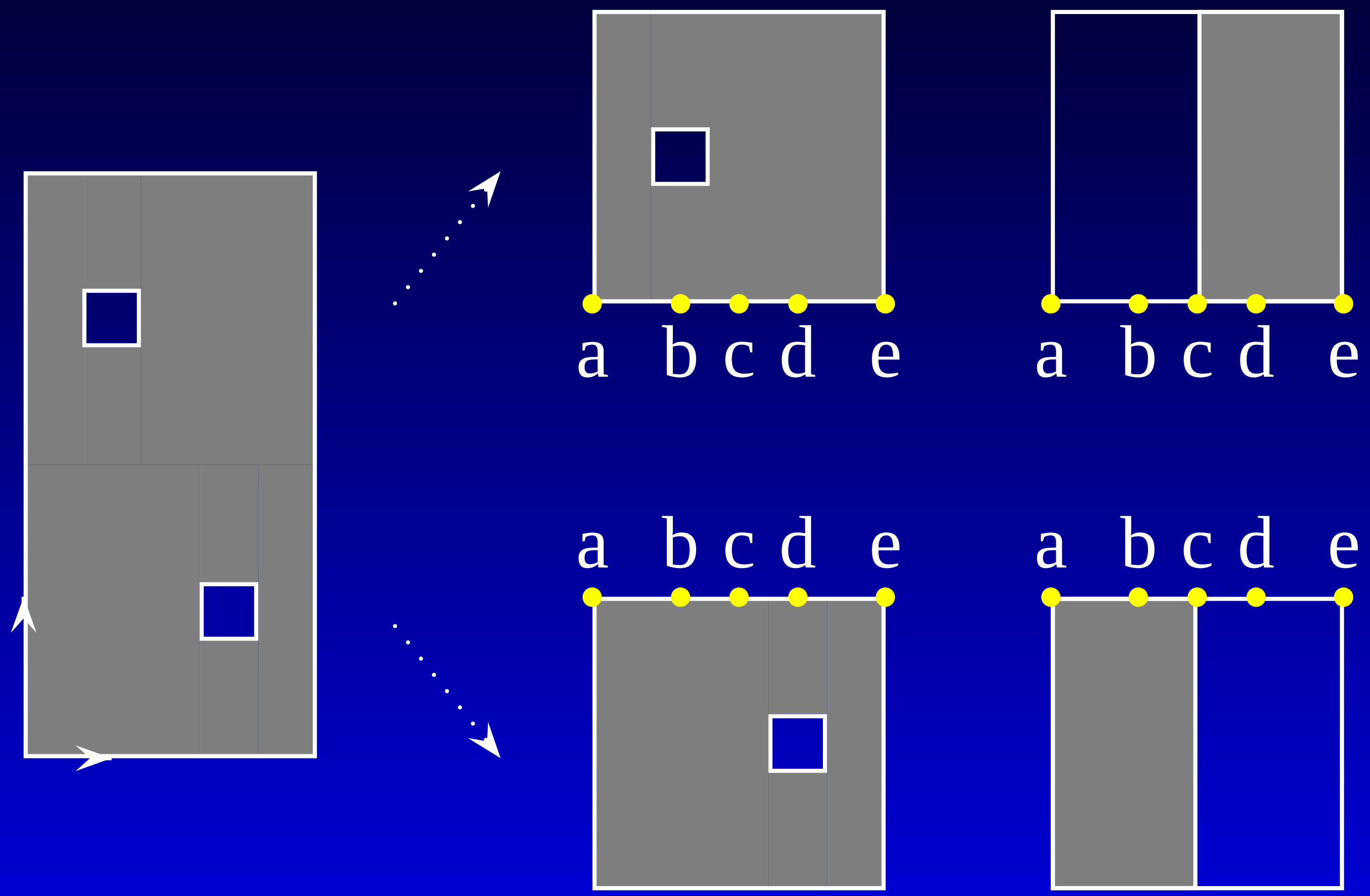
Equivalences of pieces



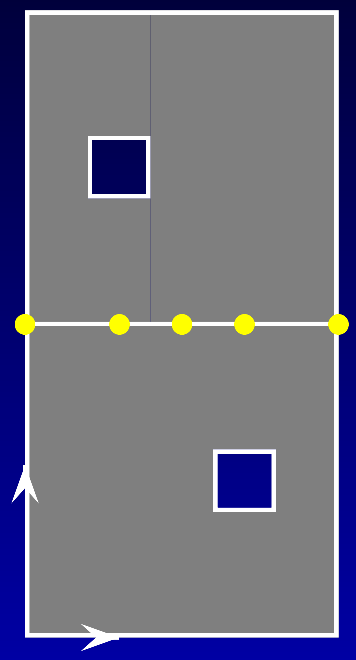
Equivalences of pieces



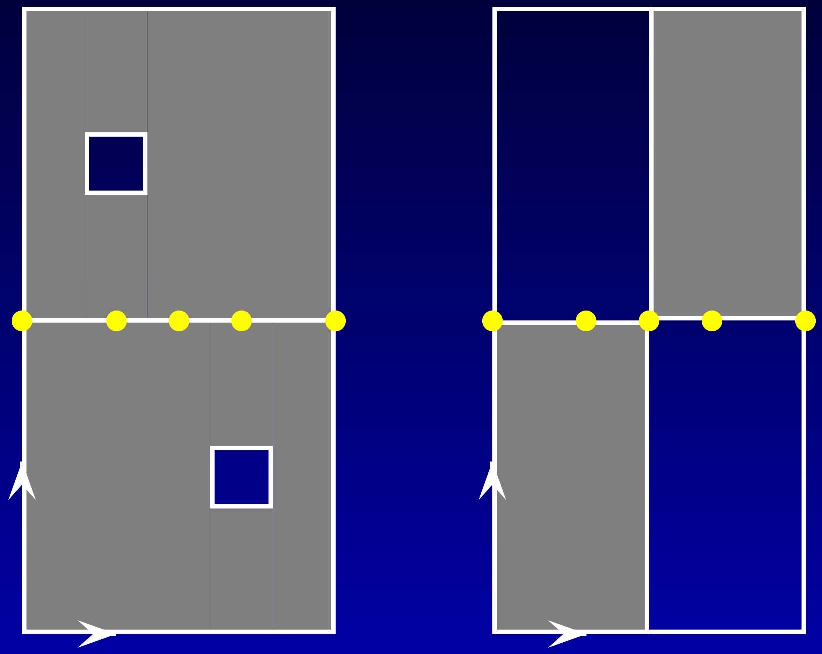
Equivalences of pieces



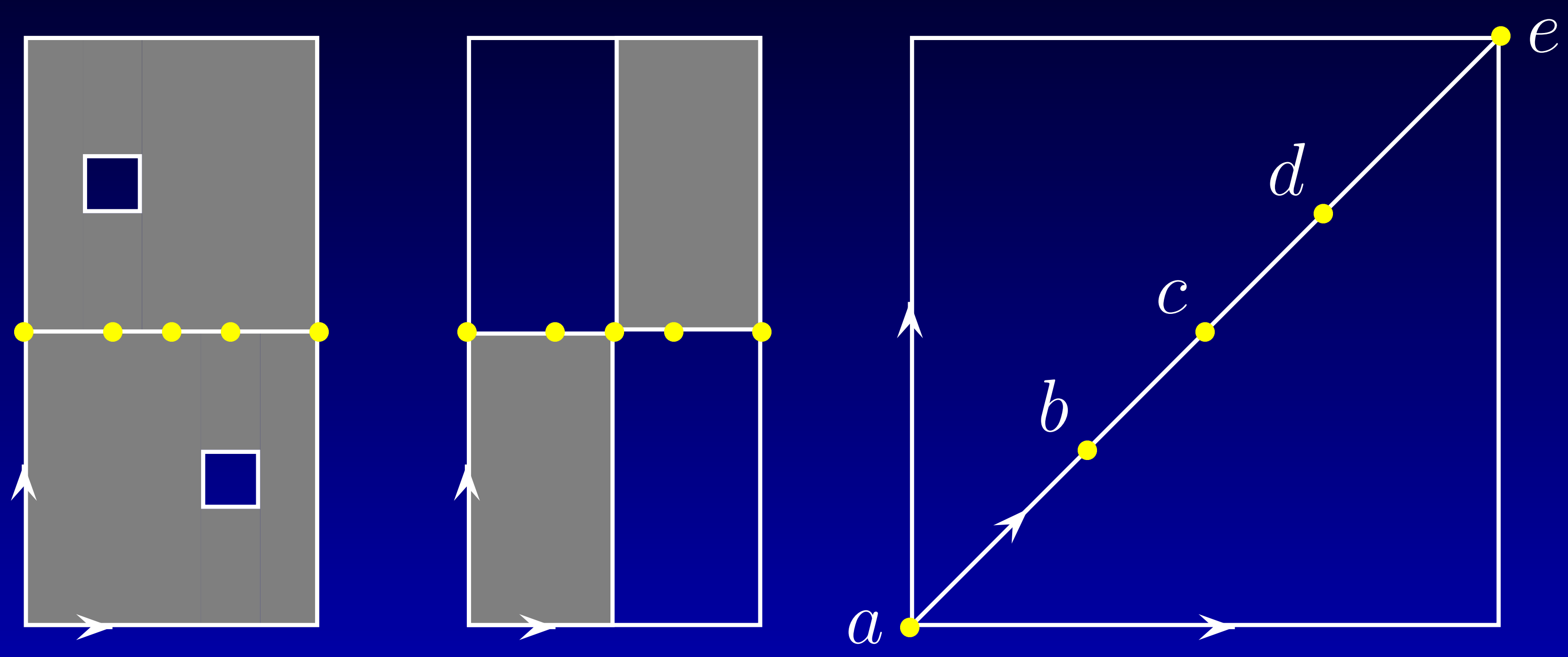
Pasting the pieces together



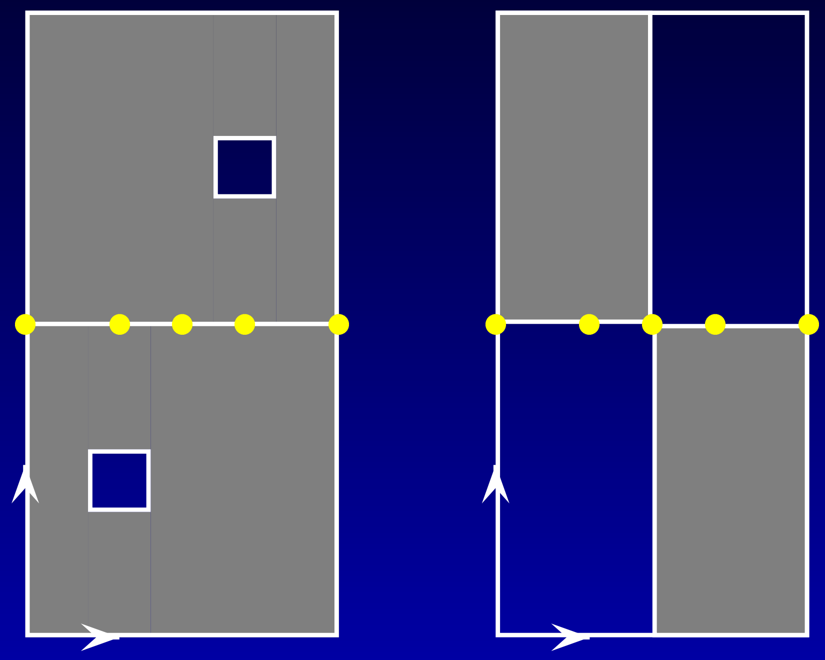
Pasting the pieces together



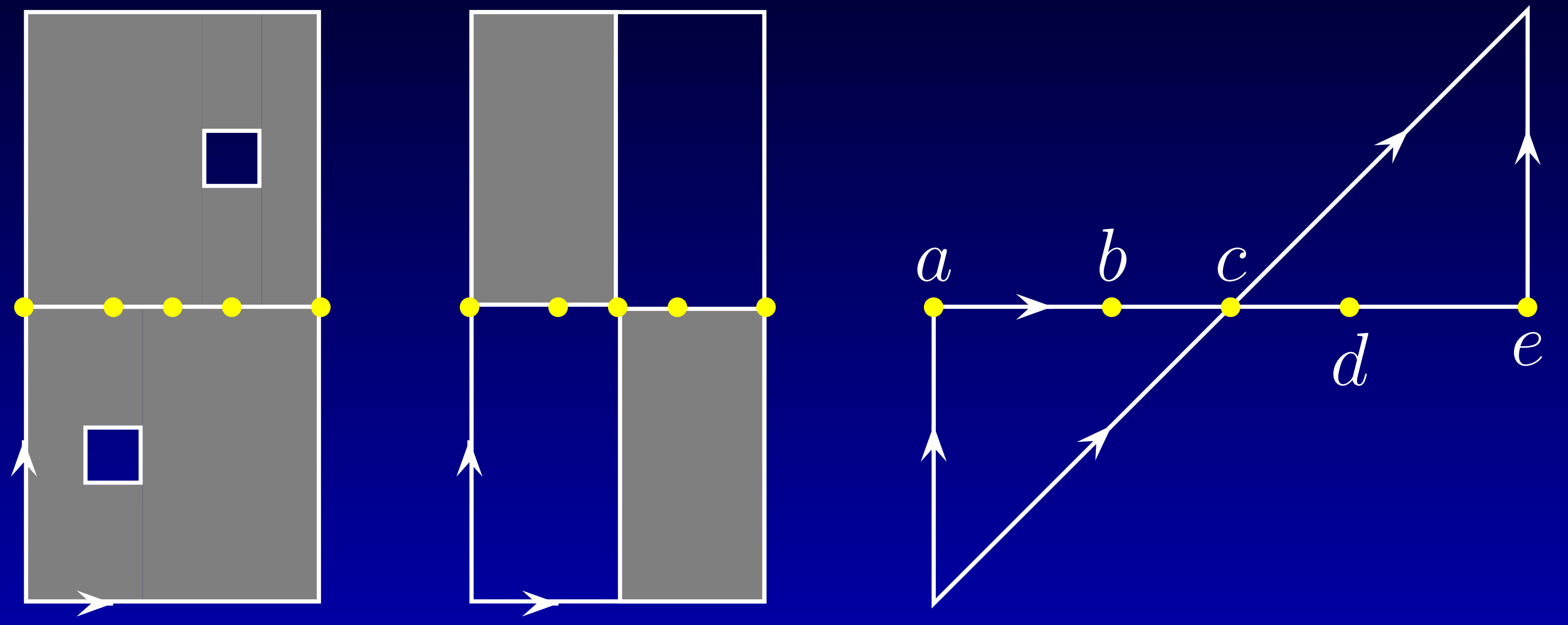
Pasting the pieces together



Second example

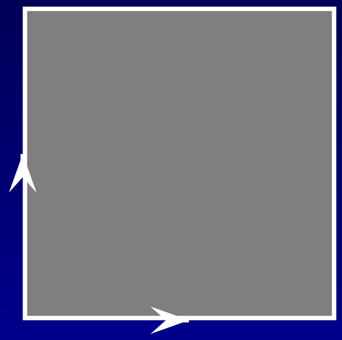


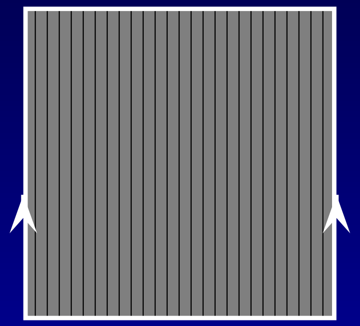
Second example



Non-discrete context

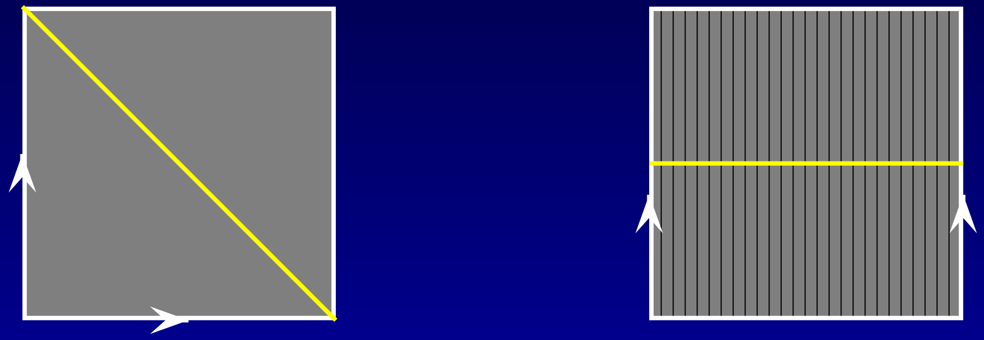
Take $\vec{I} \times \vec{I}$ and $I \times \vec{I}$





Non-discrete context

Take $\vec{I} \times \vec{I}$ and $I \times \vec{I}$



and glue them together along the yellow lines.

Conclusion

- It would be useful to have a robust notion of equivalence in models of concurrency.
- To allow a piece-by-piece analysis we would like equivalences that accept pastings.
- Using context provides such equivalences.