An introduction to directed homotopy theory

Peter Bubenik

Cleveland State University
http://academic.csuohio.edu/bubenik_p/

AMS Sectional Meeting, Kalamazoo, MI
October 19, 2008
Directed spaces

Definition

A **directed space** is a topological space $X$ together with a set $dX$ of continuous maps $[0, 1] \to X$ called **directed paths** satisfying the following:

1. all constant paths are directed paths;
2. directed paths are closed under concatenation; and
3. if $\gamma$ is a directed path and $f : [0, 1] \to [0, 1]$ is a non-decreasing continuous map then $\gamma \circ f$ is a directed path.

A **directed map** $f : (X, dX) \to (Y, dY)$ is a continuous map $f : X \to Y$ such that $dX \subseteq dY$. 

Peter Bubenik
Directed homotopy theory
Examples of directed spaces

- Any topological space $X$ is a directed space with $dX$ equal to the set of all paths in $X$. 
Examples of directed spaces

- Any topological space $X$ is a directed space with $dX$ equal to the set of all paths in $X$.
- Let $\vec{I}$ be $[0, 1]$ together with all non-decreasing continuous maps $f : [0, 1] \rightarrow [0, 1]$.
- Let $\vec{S}^1$ be the unit circle together with all counterclockwise paths.
Examples of directed spaces

- Any topological space $X$ is a directed space with $dX$ equal to the set of all paths in $X$.
- Let $\vec{I}$ be $[0, 1]$ together with all non-decreasing continuous maps $f : [0, 1] \rightarrow [0, 1]$.
- Let $\vec{S}^1$ be the unit circle together with all counterclockwise paths.
- Given two directed spaces $X$ and $Y$, then $X \times Y$ is a directed space with $d(X \times Y) = dX \times dY$ where $(f, g)(t) = (f(t), g(t))$.
- If $X$ is a directed space and $A \subseteq X$, then $A$ is a directed space with $dA$ equal to the subset of paths in $dX$ whose image is in $A$. 
Concurrent parallel computing

Several processes with shared resources

Process 1

Process 2

... 

Process N
The Swiss flag

Example

\[ \begin{align*}
V_b & \rightarrow V_a \\
Pa & \rightarrow Pb \\
Va & \leftarrow Vb \\
Pa & \rightarrow Vb
\end{align*} \]
The Swiss flag

Example
The Swiss flag

Example

Problem: Uncountably many states and execution paths.
A homotopy between directed maps $f, g : B \to C$ is a directed map $H : B \times \vec{I} \to C$ restricting to $f$ and $g$. Write $H : f \simto g$.

Directed maps $f, g$ are homotopic if there is a chain of homotopies

\[
f \simto f_1 \simto f_2 \simto \ldots \simto f_n \simto g.
\]
Equivalence classes of directed paths

Definition

Directed paths are homotopy equivalent if they are so relative to their endpoints.

Remark

Directed paths up to homotopy are significantly different from paths up to homotopy!
Directed paths

There are paths which are not homotopic to directed paths.
Two directed paths which are homotopic as paths, but not as directed paths.
The fundamental category

**Definition**

The fundamental category \( \vec{\pi}_1(X) \) has
- objects: the points in \( X \)
- morphisms: homotopy classes of directed paths

**Remark**

The existence of composition with associativity and identity is built into the definition of a category.
Problem

The fundamental category is enormous.

Plan

We would like to derive a “small” category from the fundamental category that still contains useful information.
Full subcategories of the fundamental category

Problem

The fundamental category is enormous.

Plan

We would like to derive a “small” category from the fundamental category that still contains useful information.

Definition

Given $A \subseteq X$, let $\vec{\pi}_1(X, A)$ have

- objects: points in $A$
- morphisms: homotopy classes of paths in $X$
Definition

A future retract of $\pi_1(X)$ moves each $x \in X$ along a directed path in $X$ to a point $x^+$ which “has the same future”.

$$P^+ : \pi_1(X) \to \pi_1(X, A)$$

$$\forall a \in A \exists! x \in X$$

$$[\gamma_x]$$
For $A \subseteq B \subseteq X$, one can similarly define future retracts

$$P^+ : \vec{\pi}_1(X, B) \to \vec{\pi}_1(X, A).$$

This functor is a left adjoint to the inclusion functor.

Dually, one has past retracts.
Definition (B)

An extremal model is a chain of future retracts and past retracts

$$\bar{\pi}_1(X) \xrightarrow{P_1^+} \bar{\pi}_1(X, X_1) \xrightarrow{P_2^-} \bar{\pi}_1(X, X_2) \xrightarrow{P_3^+} \ldots \xrightarrow{P_n^\pm} \bar{\pi}_1(X, A),$$

such that $\text{Min}(X) \cup \text{Max}(X) \subseteq A.$
Let $X$ be a path–connected space with $dX$ all paths. Choose $x \in X$.

There is a unique functor $\vec{\pi}_1(X) \to \vec{\pi}_1(X, x)$.

This functor is a future retract, a past retract, and a minimal extremal model.

It coincides with the functor from the fundamental groupoid to the fundamental group.
An extremal model for the Swiss flag

- a
- b
- c
- d

- a
- b
- c
- d
An extremal model of $\tilde{S}^1$

Let $x \in \tilde{S}^1$.

There is a future retract

$$P^+ : \tilde{\pi}_1(\tilde{S}^1) \to \tilde{\pi}_1(\tilde{S}^1, x) \cong (\mathbb{N}, +).$$

It is a minimal extremal model.
Van Kampen Theorem for the fundamental category

Theorem (Grandis 2003, Goubault 2003)

Assume \( X = \text{Int}(X_1) \cup \text{Int}(X_2) \) and let \( X_0 = X_1 \cap X_2 \). Then the pushout of directed spaces:

\[
\begin{array}{ccc}
X_0 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & X
\end{array}
\]

induces a pushout of fundamental categories:

\[
\begin{array}{ccc}
\vec{\pi}_1(X_0) & \longrightarrow & \vec{\pi}_1(X_1) \\
\downarrow & & \downarrow \\
\vec{\pi}_1(X_2) & \longrightarrow & \vec{\pi}_1(X)
\end{array}
\]
Compatible triples

\[ A_0 \subseteq A \subseteq X \]

\[ A_1 \subseteq B \subseteq X \]

\[ A_2 \]

\[ B_0 \]

\[ B_1 \]

\[ B_2 \]

\[ X_0 \]

\[ X_1 \]

\[ X_2 \]

\[ X \]

Peter Bubenik  Directed homotopy theory
Compatible retracts

\[
\begin{align*}
\vec{\pi}_1(X_0, A_0) & \xrightarrow{P_0^+} \vec{\pi}_1(X_0, B_0) \\
\vec{\pi}_1(X_1, A_1) & \xleftarrow{P_1^+} \vec{\pi}_1(X_1, B_1) \\
\vec{\pi}_1(X_2, A_2) & \xrightarrow{P_2^+} \vec{\pi}_1(X_2, B_2)
\end{align*}
\]
Theorem (B)

The inclusions above induce the following pushout in the arrow category on \textbf{Cat}.

\[
\begin{array}{c}
\pi_1(X_1, A_1) \quad \pi_1(X_1, B_1) \quad \pi_1(X_0, A_0) \quad \pi_1(X_0, B_0) \\
\pi_1(X, A) \quad \pi_1(X, B) \\
\pi_1(X_1, A_1) \quad \pi_1(X_1, B_1) \quad \pi_1(X_0, A_0) \quad \pi_1(X_0, B_0) \\
\pi_1(X_2, A_2) \quad \pi_1(X_2, B_2) \\
\end{array}
\]
A van Kampen theorem for future (past) retracts

Theorem (B)

There is an induced retraction $P^+$, which is a pushout.

$$
\begin{align*}
\pi_1(X_0, A_0) &\longrightarrow \pi_1(X_2, A_2) \\
\pi_1(X_1, A_1) &\longrightarrow \pi_1(X, A) \\
\pi_1(X_0, B_0) &\longrightarrow \pi_1(X_2, B_2) \\
\pi_1(X_1, B_1) &\longrightarrow \pi_1(X, B)
\end{align*}
$$
A van Kampen theorem for extremal models

Theorem (B)

The pushout of compatible extremal models is an extremal model.
Van Kampen for extremal models example

$A \subseteq B \subseteq X$
Van Kampen for extremal models example
Van Kampen for extremal models example

Peter Bubenik
Directed homotopy theory
Directed spaces provide a good mathematical model for concurrent parallel computing.

Directed paths up to homotopy are different from paths up to homotopy.

The homotopy classes of directed paths assemble into the fundamental category.

Minimal extremal models provide a way to generalize the fundamental group to directed spaces.

There is a van Kampen theorem for extremal models.
L. Fajstrup, E. Goubault, and M. Raussen (1998) used geometry and directed topology to give an algorithm for detecting deadlocks, unsafe regions and inaccessible regions for po-spaces such as the Swiss flag, in any dimension.

E. Goubault and E. Haucourt (2005) reduced the fundamental category to “components” to develop a static analyzer (ALCOOL) of concurrent parallel programs.
Open problems

- The fundamental bipartite graphs detects deadlocks and captures the essential schedules. Is this part of a homology theory?
- What can we do with higher directed homotopy?