A mathematical model for concurrent parallel computing

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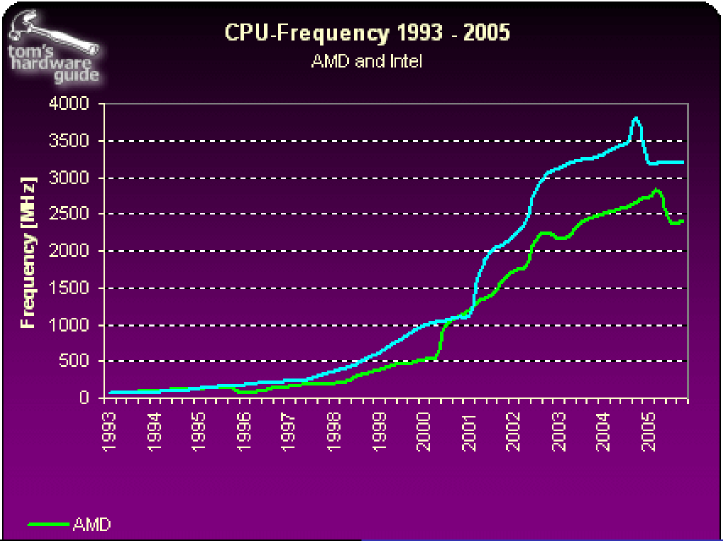
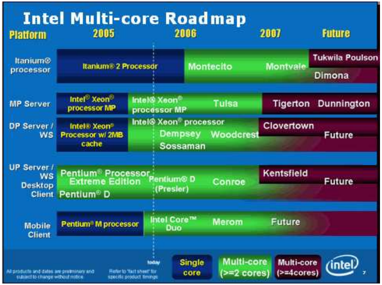
The end of Moore's Law?

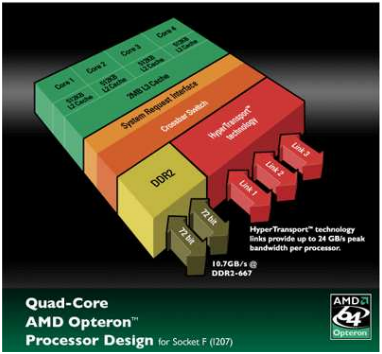


Example: Multi-core processors

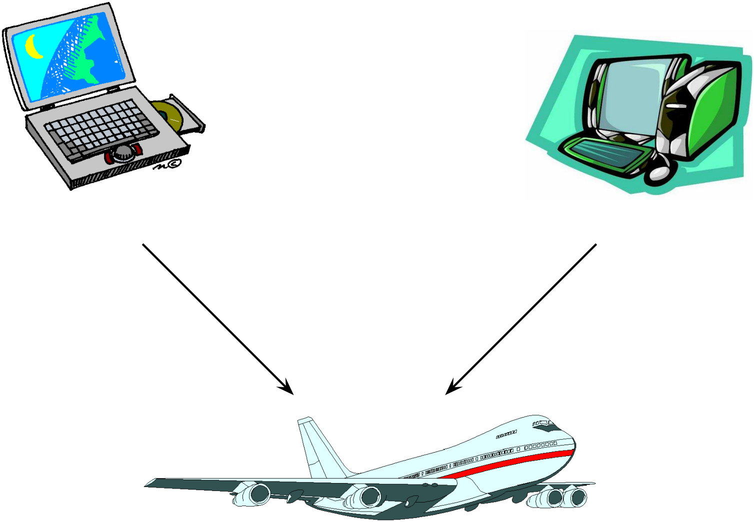
Intel



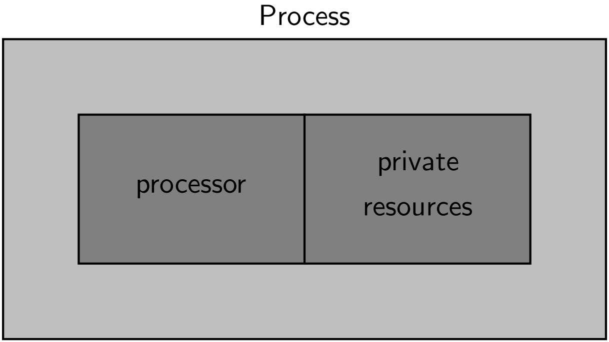
AMD



Example: Internet database

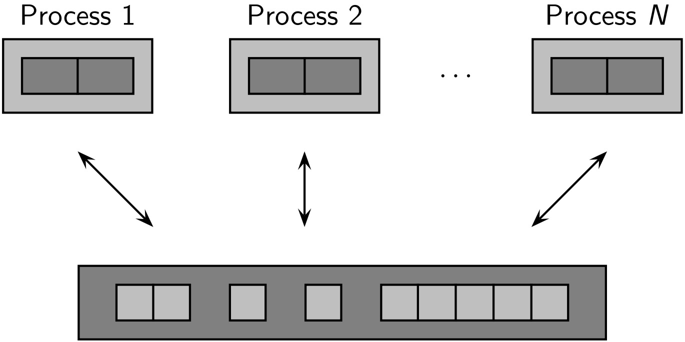


Classical non-parallel computing



A process with its own private resources

Concurrent parallel computing



Several processes with shared resources

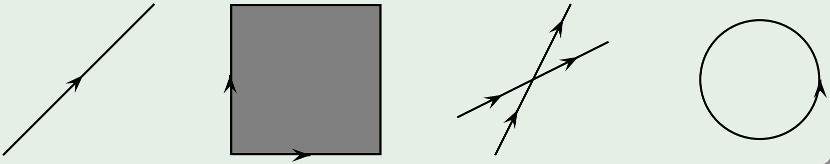
A mathematical model

Concurrent systems can be modeled by spaces in which only some certain paths are allowed.

Definition

A directed space is a topological space X with a distinguished set of paths, $\gamma : [0,1] \to X$, called directed paths.

Example



A concurrent system

Example

2 processes using 2 shared resources *a* and *b* which can only be used by one process at a time

Notation

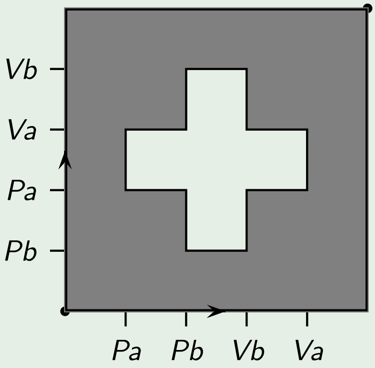
Px - a process locks resource x
Vx - a process releases resource x

Program

The first process:	*Pa*	*Pb*	*Vb*	*Va*
The second process:	*Pb*	*Pa*	*Va*	*Vb*

The Swiss flag

Example



The Swiss flag

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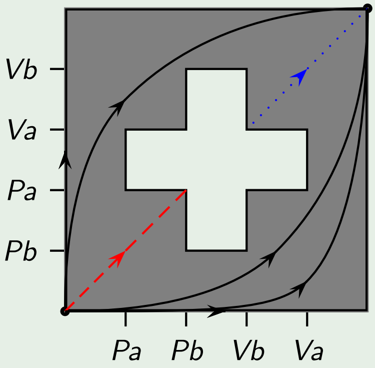

The Swiss flag

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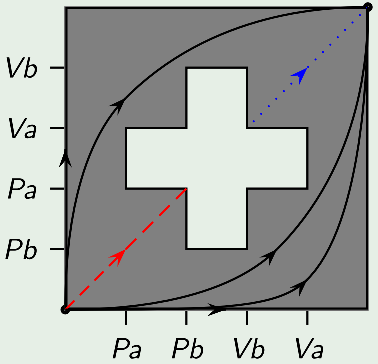
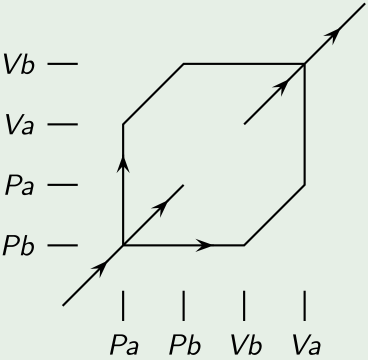


Problem: The state space is infinite.

The essential schedules of the Swiss flag

Example



This is a subspace of the Swiss flag.

Goal

Develop a framework for concurrency where equivalences are accounted for.

We would like equivalences that allow a piece-by-piece analysis. This will make the analysis of large programs tractable.

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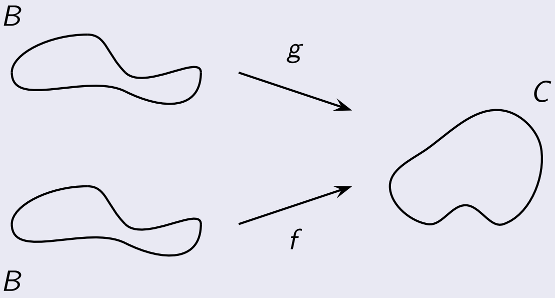
Idea

Use algebraic topology.

Undirected equivalences

Definition

Given continuous maps $f, g : B \to C$,



Undirected equivalences

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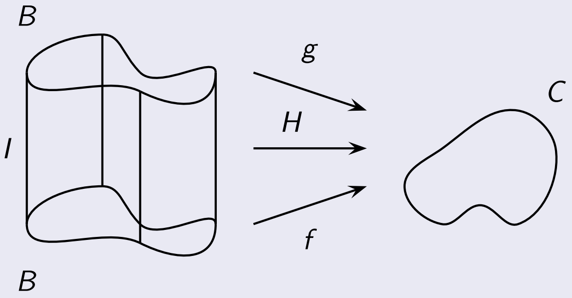

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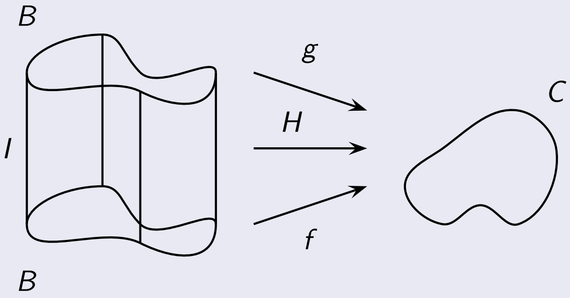


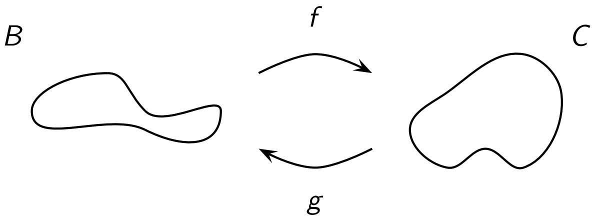
a homotopy between f and g is a continuous map $H : B \times I \to C$ restricting to f and g. This is an equivalence relation. Write $H : f \cong g$

Undirected equivalences

Definition

Spaces B, C are homotopy equivalent if there are maps $f : B \leftrightarrows C : g$ such that

$$g \circ f \simeq \mathrm{Id}_B \text{ and } f \circ g \simeq \mathrm{Id}_C .$$



Directed spaces

Definition

- A directed space is a topological space X with a distinguised set of directed paths.
- A directed map is a continuous map $f : X \to Y$ between directed spaces such that if $\gamma : [0,1] \to X$ is a directed paths in X, then $f(\gamma)$ is a directed path in Y.

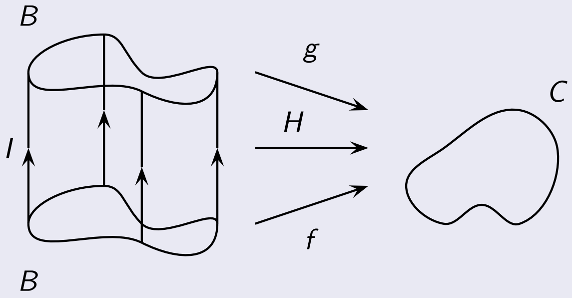
Remark

Subspaces, products, and quotients of directed spaces inherit a directed space structure.

Directed equivalences

Definition

- A homotopy between directed maps $f, g : B \to C$ is a directed map $H : B \times \vec{I} \to C$ restricting to f and g. Write $H : f \to g$.



Directed equivalences

Definition

- Write $f \simeq g$ if there is a chain of dihomotopies

$$f \longrightarrow f_1 \longleftarrow f_2 \longrightarrow \ldots \longleftarrow f_n \longrightarrow g.$$

- Po-Spaces B, C are dihomotopy equivalent if there are dimaps $f : B \leftrightarrows C : g$ such that

$$g \circ f \simeq \mathrm{Id}_B \ \text{ and } f \circ g \simeq \mathrm{Id}_C \, .$$

The fundamental category

Definition

The fundamental category $\vec{\pi}_1(X)$ has

- objects: the points in X
- morphisms: homotopy classes of directed paths

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- objects: the points in X
- morphisms: homotopy classes of directed paths

Problem

The fundamental category is enormous.

Full subcategories of the fundamental category

Plan

We would like to derive a "small" category from the fundamental category that still contains useful information.

Definition

Given $A \subseteq X$, let $\vec{\pi}_1(X, A)$ have

- objects: points in A
- morphisms: homotopy classes of paths in X

The fundamental bipartite graph

Definition

For (X, dX) write $x \leq y$ if there exists a dipath γ with $\gamma(0) = 1$ and $\gamma(1) = y$. This gives X a preorder.

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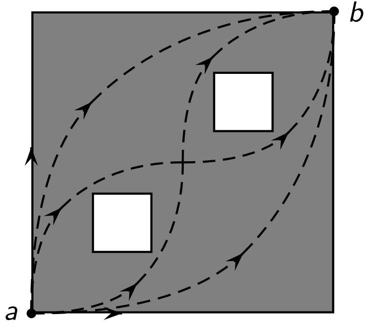
Definition

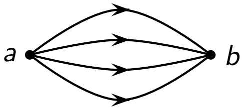
Let $\mathrm{Min}(X) = \{a \in X \mid a' \leq a \implies a' = a\}$.
Let $\mathrm{Max}(X) = \{b \in X \mid b \leq b' \implies b = b'\}$.

Definition (B)

The fundamental bipartite graph of X is $\vec{\pi}_1(X, \mathrm{Min}(X) \cup \mathrm{Max}\, X)$.

Example of the fundamental bipartite graph

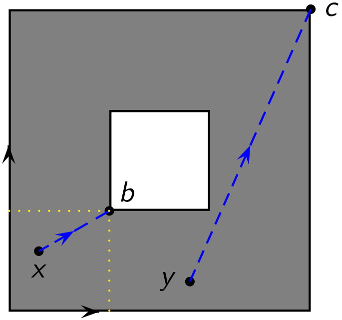


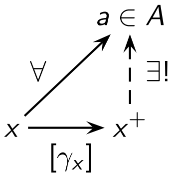


Future retracts

Definition

A future retract of $\vec{\pi}_1(X)$ moves each $x \in X$ along a directed path in X to a point x^+ which "has the same future".



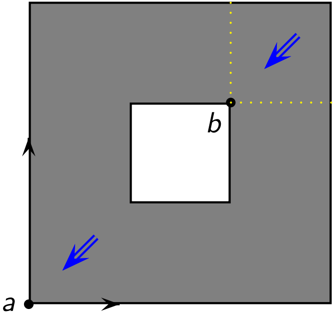


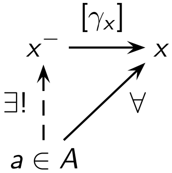
$$P^+ : \vec{\pi}_1(X) \to \vec{\pi}_1(X, A)$$

Past retracts

Definition

A past retract of $\vec{\pi}_1(X)$ moves each $x \in X$ backwards along a directed path in X to a point x^- which "has the same past".





$$P^- : \vec{\pi}_1(X) \to \vec{\pi}_1(X, A)$$

Extremal models

Definition (B)

An extremal model is a chain of future retracts and past retracts

$$\vec{\pi}_1(X) \xrightarrow{P_1^+} \vec{\pi}_1(X, X_1) \xrightarrow{P_2^-} \vec{\pi}_1(X, X_2) \xrightarrow{P_3^+} \dots \xrightarrow{P_n^\pm} \vec{\pi}_1(X, A),$$

such that $\mathrm{Min}(X) \cup \mathrm{Max}(X) \subseteq A$.

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such that $\mathrm{Min}(X) \cup \mathrm{Max}(X) \subseteq A$.

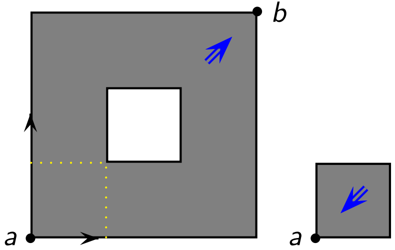
Proposition (B)

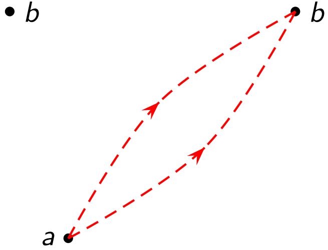
An extremal model induces an injection of fundamental bipartite graphs.

Theorem (B)

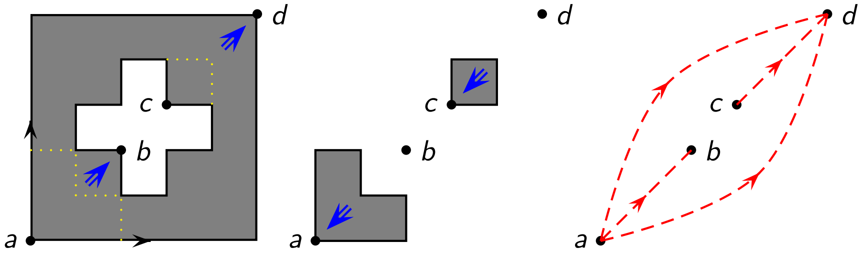
If X is a compact and $\leq$ is a partial order, then extremal models induce an isomorphism of fundamental bipartite graphs.

Examples of extremal models

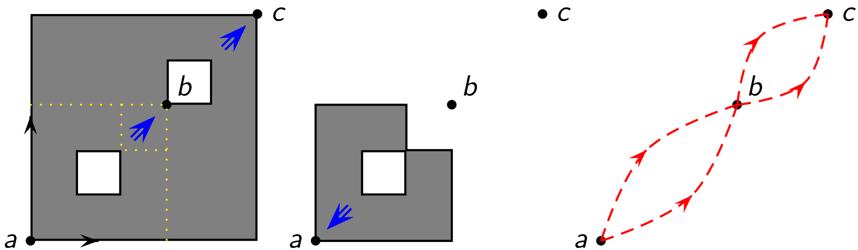




An extremal model for the Swiss flag



Examples of extremal models



An extremal model of $\vec{S}^1$

Let $x \in \vec{S}^1$.

There is a future retract

$$P^+ : \vec{\pi}_1(\vec{S}^1) \longrightarrow \vec{\pi}_1(\vec{S}^1, x) \cong (\mathbb{N}, +).$$

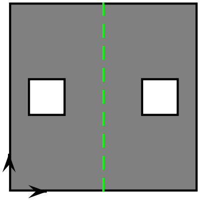
It is a minimal extremal model.

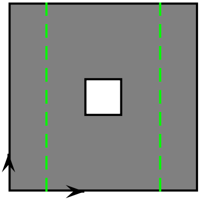
Piecewise analysis

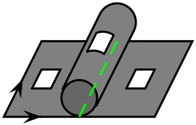
Theorem (B)

Extremal models can be constructed in a piecewise manner.

Van Kampen for extremal models example







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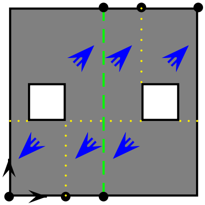

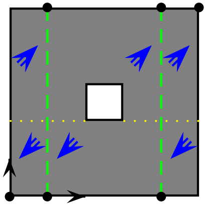

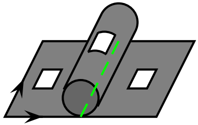

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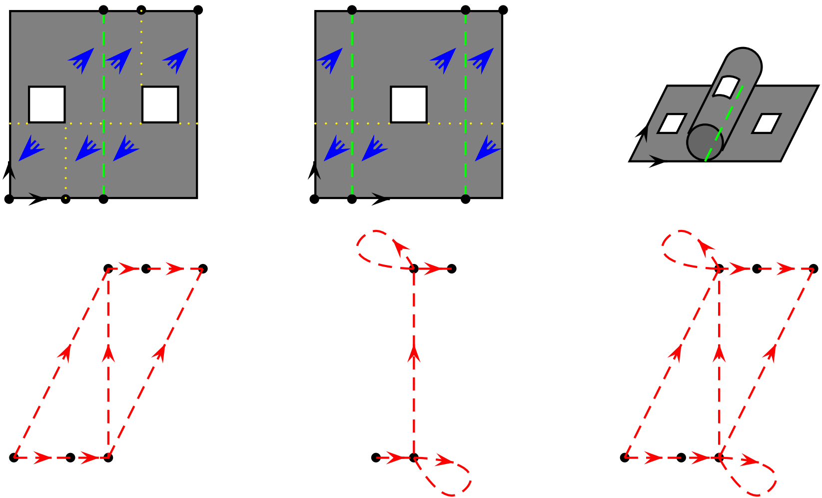

Summary

- Directed spaces provide a good mathematical model for concurrent parallel computing.
- Using directed homotopies one can hope to cope with the "state space explosion".
- The homotopy classes of directed paths assemble into the fundamental category.
- Minimal extremal models provide a way to generalize the fundamental group to directed spaces.
- Extremal models can be constructed in a piecewise manner.

Applications

- L. Fajstrup, E. Goubault, and M. Raussen (1998) used geometry and directed topology to give an algorithm for detecting deadlocks, unsafe regions and inaccessible regions for po-spaces such as the Swiss flag, in any dimension.
- E. Goubault and E. Haucourt (2005) reduced the fundamental category to "components" to develop a static analyzer (ALCOOL) of concurrent parallel programs.