Finite Element Approximation for Grad-Div Type Systems in the Plane

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FINITE ELEMENT APPROXIMATION FOR 
GRAD-DIV TYPE SYSTEMS IN THE PLANE*

CHING LUNG CHANG†

Abstract. This paper studies a 3 x 3 linear system of differential equations of grad-div type with appropriate boundary conditions. In constructing numerical approximate solutions to this boundary value problem it applies a least squares finite element method to an extended system of four equations and three unknowns. Analysis of the rate of convergence of the numerical solutions and estimates of the error involved in approximating exact solutions of the considered boundary value problems are given. The method of this paper achieves optimal rates of convergence both in the $H^1$-norm and in the $L^2$-norm. Numerical examples employing piecewise-linear elements for directional triangulation and bilinear elements in a quadrilateral grid are also presented.

Key words. Sobolev spaces, a priori estimate, least squares method, Fredholm operator, $\| \cdot \|_{-1}$ norm

AMS(MOS) subject classifications. 65N30, 35F15

1. Introduction. This paper treats both theoretical and computational aspects of the problem of solving a $3 \times 3$ linear system in two variables of the form $\text{grad}(g) = u$ and $\text{div} \ u = f$. The classical finite element method is used for Poisson’s equation $\Delta g = f$, but there may be a less accurate approximation to its gradient $\text{grad}(g)$. In the solution of problems concerning fluid mechanics or elasticity the values of the derivatives of $g$ are sought rather than the values of $g$ itself.

The least-squares method and standard Galerkin finite element method can lead to stable numerical results, provided we make suitable assumptions regarding the shapes of the elements. For instance, under the grid decomposition property as a condition, in [8] an optimal order estimate for piecewise polynomial approximation spaces is derived. In [11], the authors derive an appropriate variational form for the Poisson equation, where the intent is to solve for the function and its gradient. This method performs well, given properly chosen restrictions on the finite element space.

By contrast, this paper employs a more standard least-squares finite element approximation, based on an extended system which becomes a regular elliptic boundary value problem [12]. A so-called “slack” variable $k$ is introduced; it is needed in the proof of convergence and in the error analysis; the slack variable does not play a role in our computations. We apply the least-squares method to a system with three variables and four equations. Analysis of our algorithms shows that they yield optimal rates of convergence, in both the $H^1$ and $L^2$ norms, for computing either $g$ or $\nabla g$ in the finite-dimensional subspace of continuous piecewise polynomials of degree $r$. For example, if $r = 1$, i.e., if a continuous piecewise linear function or piecewise bilinear function is assumed, then the resulting approximations $g^h$ and $u^h$ satisfy

$$\| g - g^h \|_0 + h \| g - g^h \|_1 \leq C h^2 \| g \|_2$$
and
\[ \|u - u^h\|_0 + h\|u - u^h\|_1 \leq C h^2 \|u\|_2. \]

Throughout this paper \( C > 0 \) is a constant independent of \( h \) with different values in different places in the paper. Our results hold without any restriction other than those stated and the regularity requirements for the triangulation (cf. [4]). The numerical examples given in the case of \( H^1 \) spaces with respect to piecewise linear elements in directional triangles and piecewise bilinear elements in quadrilaterals support our theoretical analysis. In some sense we can say that we have found a "superconvergent" approximation to a solution of the Poisson equation, since with piecewise linear element the \( L^2 \) errors both in \((g - g^h)\) and in \((u - u^h)\) are \( O(h^2) \).

2. Some notation and the formulation of the problem. The two-dimensional Poisson equation can be transformed into a first-order linear system by regarding the function itself and the components of the gradient as variables. Consequently, we obtain the grad-div \( 3 \times 3 \) linear system:

\[
\begin{align*}
\text{grad}(g) - u &= 0 \quad \text{in } \Omega, \\
\text{div} \ u &= f \quad \text{in } \Omega, \\
Ru &= 0 \quad \text{on } \Gamma.
\end{align*}
\]

In the above, we assume that \( \Omega \) is a simply connected bounded domain in \( \mathbb{R}^2 \) with a Hölder continuously differentiable, positively oriented boundary \( \Gamma \). We also assume that \( f \) is a smooth function defined on \( \Omega \). To simplify the numerical scheme and the analysis, we actually assume that \( \Omega \) is a convex polygon in \( \mathbb{R}^2 \). The boundary conditions are assumed to be of the form \( Ru = n \times u \) or \( Ru = n \cdot u \); these conditions are associated with the homogeneous Dirichlet and Neumann boundary conditions of the Poisson equation, respectively. In the case \( Ru = n \cdot u \), \( f \) has to satisfy the compatibility condition \( \int_\Omega f = 0 \).

Throughout, we will employ standard notations from the theory of Sobolev spaces, including the usual norms. In particular, we let \( H^s(\Omega) \) denote the Sobolev space of functions which have square integrable derivatives of order up to \( s \) on \( \Omega \); we write \( \| \cdot \|_s \) for the standard Sobolev norm associated with \( H^s(\Omega) \). \( H^s_0(\Omega) \) denotes the closure of \( \mathcal{D}(\Omega) \) with respect to the norm \( \| \cdot \|_s \). \( H^{-s}(\Omega) \) denotes the dual space of \( H^s_0(\Omega) \) and is given the structure of a normed linear space by setting, for any \( \rho \in H^{-s}(\Omega) \),

\[ \|\rho\|_{-s} = \sup \frac{|(\rho, \varphi)|}{\|\varphi\|_s}, \]

the sup being over the set of all \( \varphi \in H^s_0(\Omega) \) \( \neq 0 \). For subsets \( \Omega^h \subset \Omega \) we may also consider the Sobolev space \( H^s(\Omega^h) \) together with its associated norm \( \| \cdot \|_{s, \Omega^h} \).

For any \( \Omega, \Gamma, \) and \( s \) as above let

\[
V^s = \left\{ \begin{pmatrix} v \\ \rho \end{pmatrix} \in [H^s(\Omega)]^3 ; \quad Rv = 0 \text{ on } \Gamma \right\},
\]

(2.2)

\[
V' = \left\{ \begin{pmatrix} v \\ \rho \\ k \end{pmatrix} \in [H^s(\Omega)]^4 ; \quad \begin{pmatrix} v \\ \rho \end{pmatrix} \in V^s, \quad k \in H^s_0(\Omega) \right\}.
\]

Next let \( \tau_h \) denote a family of regular triangulations [4] of the polygonal domain \( \Omega^h \) into triangles \( \Omega_j, j = 1, 2, \cdots, T \), where \( h \) is the maximum diameter of these
triangles. Let $P_r(\Omega_j)$ denote the space of polynomials of degree less than or equal to $r$ defined on $\Omega_j$, and let

$$S^h_j = \left\{ \begin{pmatrix} \nu^h_j \\ \rho^h \end{pmatrix} \in [H^1(\Omega)]^3; \; v^h_1, v^h_2 \text{ and } \rho^h \in P_r(\Omega_j) \; j = 1, 2, \ldots, T \right\}$$

and

$$V^h = S^h_1 \cap V^1.$$

Note that $S^h_j \subset C^0(\Omega)$ and $V^h_j$ satisfies the approximation property [2], [4]: for $t = 0, 1$ and every $(\nu^s) \in [H^s(\Omega)]^3$, $1 \leq s \leq r + 1$, there exists a $(\nu^s_\rho) \in V^h_j$ such that

$$\| \nu - \nu^h \|_t + \| \rho - \rho^h \|_t \leq C_{s,t} h^{s-t} (\| \nu \|_s + \| \rho \|_s),$$

the positive constants $C_{s,t}$ being independent of $\nu$, $\rho$, and $h$.

3. The extended system and least squares finite element method. Since the $3 \times 3$ linear grad-div system is not elliptic in the sense of Petrovski (even though it is elliptic in the sense of Dougilis and Nirenberg [6]), the least squares method applied directly to (2.1) may lead to an unstable result. To avoid this problem, we extend (2.1) by setting

$$L(u, g) = \begin{bmatrix} \text{grad}(g) - u \\ \text{div} u \\ \text{curl} u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} = F,$$

where $\text{curl} u = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}$.

We consider the simplest formulation of our problem such that (3.1), with the boundary condition $Ru = 0$, has exactly one solution and is also solvable for an $f$ which satisfies the compatibility condition.

Let us define a quadratic functional

$$J(\nu, \rho) = \| L(\nu, \rho) - F \|_0^2 \text{ for } \begin{pmatrix} \nu \\ \rho \end{pmatrix} \in V^1.$$

It is easily seen that a solution of (2.1) minimizes (3.2) and that a sufficiently smooth solution of (3.1) also satisfies (2.1).

For arbitrary $(\nu^h_\rho) \in V^h_j$, as in (2.3),

$$\| L(\nu^h, \rho^h) - F \|_0^2$$

will not vanish in general. The choice of $(u^h, g^h) \in V^h_j$ which results in a minimum value for (3.3) depends on the metric chosen for (3.3). In this paper we consider an $L^2$-norm; in this case the problem of minimizing (3.2) becomes a problem of Chebyshev approximation [12].

Observe that $(u^h, g^h)$ minimizes (3.2) if and only if

$$a(u^h, g^h; v^h, \rho^h) = b(v^h, \rho^h) \text{ for all } \begin{pmatrix} v^h \\ \rho^h \end{pmatrix} \in V^h_j,$$
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where

\[ a(u, g; v, \rho) = \int_{\Omega} L(u, g) \cdot L(v, \rho) \]
\[ = \int_{\Omega} (\text{grad}(g) - u) \cdot (\text{grad}(\rho) - v) + \text{div } u \text{ div } v + \text{curl } u \text{ curl } v \]

and

\[ b(v, \rho) = \int_{\Omega} \rho \text{div } v. \]

Choosing a basis for \( V_h \) reduces (3.4) to a set of algebraic linear systems. Evidently, the corresponding matrix of this system is symmetric; it will be proved to be positive definite in the next section.

4. Convergence and error estimate. To prove that the bilinear form in (3.4) is coercive, we introduce a “slack” variable \( k \) which is smooth in \( \Omega \), vanishes on the boundary \( \Gamma \), and satisfies

\[ L'(u, g, k) = \begin{bmatrix} \text{grad } (g) - \text{curl } (k) - u \\ \text{div } u \\ \text{curl } u \end{bmatrix} = F \]

where \( \text{curl } (k) = \left( \frac{\partial k}{\partial y}, -\frac{\partial k}{\partial x} \right)^T \). The corresponding extended boundary conditions are

\[ R'(u, g, k) = \begin{bmatrix} n_2 & -n_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ g \\ k \end{bmatrix} = 0 \quad \text{on } \Gamma, \]

if \( Ru = n \times u \), and

\[ R'(u, g, k) = \begin{bmatrix} n_1 & n_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ g \\ k \end{bmatrix} = 0 \quad \text{on } \Gamma \]

if \( Ru = n \cdot u \).

Equation (4.1) can be rewritten as

\[ L'(u, g, k) = A \begin{bmatrix} u \\ g \\ k \end{bmatrix}_x + B \begin{bmatrix} u \\ g \\ k \end{bmatrix}_y + C \begin{bmatrix} u \\ g \\ k \end{bmatrix} = F = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}, \]

where

\[ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]
It is easy to verify that (4.1), (4.3) is strongly elliptic, i.e., \( \det(\xi A + \eta B) = (\xi^2 + \eta^2)^2 \), so \( \xi A + \eta B \) is a nonsingular matrix for each real pair \((\xi, \eta) \neq (0, 0)\). After elementary operations, following the procedure given by [12, pp. 76-81], we can check that the corresponding boundary condition (4.2), (4.2)' satisfies the Lopatinski condition. Thus, (4.1), (4.2) (or (4.2)') is a regular elliptic boundary value problem. It follows that the operator \((L', R')\) is a Fredholm operator, which implies that it has a finite-dimensional nullspace. Indeed, the nullity of (4.1), (4.2) (or (4.2)') is one. Let

\[
\Lambda(u, g, k) = g(x_0) = 0, \quad x_0 \text{ is a point on } \Omega.
\]

Then (4.1), (4.2) admits at most one solution. The a priori estimate for (4.1) and (4.2) can be written as: There exists \(C_p > 0\) for \(p \geq 1\) such that

\[
(4.5) \quad C_p^{-1} \left\| \begin{pmatrix} u \\ g \\ k \end{pmatrix} \right\|_p \leq \|L'(u, g, k)\|_{p-1} + \|R'(u, g, k)\|_{p-\frac{1}{2}} + |g(x_0)| \leq C_p \left\| \begin{pmatrix} u \\ g \\ k \end{pmatrix} \right\|_p.
\]

Clearly, (4.5) remains valid for \(k \equiv 0\). Furthermore, if we consider the problem with respect to \(V^1\) and let \(g(x_0) = 0\), we obtain

\[
(4.6) \quad C_p^{-1} \left\| \begin{pmatrix} u \\ g \end{pmatrix} \right\|_p \leq \|L(u, g)\|_{p-1} \leq C_p \left\| \begin{pmatrix} u \\ g \end{pmatrix} \right\|_p,
\]

since \(L'(u, g, 0) = L(u, g)\).

4.1. **H¹-norm convergence.** The coercivity of the bilinear form \(a(u, g; v, p)\) in (3.4) is implied by (4.6) when \(p = 1\). It follows that the linear algebraic system generated by (3.4) is positive definite and symmetric; we obtain the following result.

**Theorem 4.1.** For each \(h > 0\) there exists uniquely an element \((u^h, g^h) \in V_h^r\) such that \(\|L(u^h, g^h) - F\|_0^2\) assumes a minimum value in \(V_h^r\). The matrix corresponding to the numerical scheme that determines this value of \((u^h, g^h)\) is symmetric and positive definite. Moreover, there exists a constant \(C > 0\) that is independent of \(h\) such that

\[
(4.7) \quad \|u - u^h\|_1 + \|g - g^h\|_1 \leq C (\|u - v^h\|_1 + \|g - p^h\|_1) \quad \text{for any } \begin{pmatrix} v^h \\ p^h \end{pmatrix} \in V_h^r.
\]

**Corollary 4.2.** Let \((u, g)\) be the solution of (2.1) and let \((u^h, g^h) \in V_h^r\) be the solution of (3.4). If \((u, g) \in V^p\), where \(1 \leq p \leq r + 1\), then there exists a constant \(C > 0\), independent of \(h\), such that

\[
(4.8) \quad \|u - u^h\|_1 + \|g - g^h\|_1 \leq Ch^{p-1} (\|u\|_p + \|g\|_p).
\]

**Proof.** \((u, g) \in V^p\) satisfies (2.1) and \(\begin{pmatrix} u \\ g \end{pmatrix} \) must satisfy (4.1) and (4.2) in the case \(k \equiv 0\) in \(\Omega\). Combining (2.5) and (4.7), we obtain (4.8).

4.2. **L²-norm convergence.** Given the \(H^1\)-norm error estimate (4.8), the proof of the \(L^2\)-norm error estimate depends essentially on a duality argument. The inequalities (4.5) were shown by [1] and [12] to hold for the case \(p \geq 1\). Dikanski\[5\] extended these results, by an interpolation argument, to the case \(p \geq 0\). If \(p = 0\), then

\[
(4.9) \quad C_0^{-1} \left\| \begin{pmatrix} u \\ g \\ k \end{pmatrix} \right\|_0 \leq \|L'(u, g, k)\|_{-1} \leq C_0 \left\| \begin{pmatrix} u \\ g \\ k \end{pmatrix} \right\|_0.
\]
where $C_0 > 0$ is independent of $u$, $g$, and $k$.

The boundary value problem (4.1), (4.2), or (4.2)' may not be solvable for all $F \in [H^{p - 1}(\Omega)]^d$. Let $N^*$ denote the codimension of the range of $L'$ in the space $[H^{p - 1}(\Omega)]^d$. Since $(L', R')$ in (4.1) and (4.2) or (4.2)' is regular elliptic, it is a Fredholm operator. The adjoint form of $(L', R')$ satisfies an a priori estimate and $N^*$ is finite [12]. In fact, $N^*$ is the number of compatibility conditions required to ensure the existence of a solution of (4.1) and (4.2) or (4.2)'.

Therefore, generally, there exist $N^*$ linear independent orthonormal function vectors $I_j (j = 1, \ldots, N^*)$ such that

\begin{equation}
\begin{cases}
I_j(x) \in [H^{s-1}(\Omega)]^d, \\
(I_j(x), I_\ell(x)) = \delta_{j\ell}, \\
(I_j, L'(u, g, k)) = 0 \text{ for any } (u, g, k) \in V'^s.
\end{cases}
\end{equation}

The extended problem of (3.1), (3.2), or (3.2)' can be written as

\begin{equation}
\begin{cases}
L'(u, g, k) + \sum_{i=1}^{N^*} \alpha_i I_i = \Phi & \text{in } \Omega, \\
R'(u, g, k) = 0 & \text{on } \Gamma, \\
g(x_0) = 0 & \text{for an } x_0 \in \Omega;
\end{cases}
\end{equation}

it admits exactly one solution $(u, g, k; \alpha)$ for any given $\Phi \in [H^p(\Omega)]^d$. We define

\begin{equation}
J'(v, \rho, \gamma) = \|L'(v, \rho, \gamma) - \Phi\|_0^2 \text{ for } \begin{pmatrix} v \\ \rho \\ \gamma \end{pmatrix} \in V'^p, \ p \geq 1
\end{equation}

and choose $\begin{pmatrix} v_h \\ \rho_h \\ \gamma_h \end{pmatrix}$ with the property that (4.12) assumes its minimal value in $V'^h$ if and only if

\begin{equation}
a'(u^h, g^h, k^h; v^h, \rho^h, \gamma^h) = b'(v^h, \rho^h, \gamma^h) \text{ for } \begin{pmatrix} v^h \\ \rho^h \\ \gamma^h \end{pmatrix} \in V'^h,
\end{equation}

where

\begin{equation}
V'^h = \left\{ \begin{pmatrix} v^h \\ \rho^h \\ \gamma^h \end{pmatrix} \in V'^s; \begin{pmatrix} v^h \\ \rho^h \\ \gamma^h \end{pmatrix} \in V'^h \text{ and } \gamma^h \in P_r(\Omega_j), \ j = 1, 2, \ldots, T \right\},
\end{equation}

\begin{equation}
a'(u, g, k; v, \rho, \gamma) = \int_\Omega L'(u, g, k) \cdot L'(v, \rho, \gamma),
\end{equation}

\begin{equation}
b'(v, \rho, \gamma) = \int_\Omega \rho \text{ div } v.
\end{equation}

The coercivity of $a'(u, g, k; u, g, k)$ follows by (4.5). We also have the orthogonality relation

\begin{equation}
a'(u - u^h, g - g^h, k - k^h; v^h, \rho^h, \gamma^h) = 0 \text{ for any } \begin{pmatrix} v^h \\ \rho^h \\ \gamma^h \end{pmatrix} \in V'^h.
\end{equation}
For any given \( \Phi \in [H^{s-1}(\Omega)]^4 \), there is a unique \((w, \phi, \psi)^T \in V^r\), \(\alpha \in \mathbb{R}^{N^*}\) such that

\[
\begin{cases}
L'(w, \phi, \psi) + \sum_{i=1}^{N^*} \alpha_i I_i = \Phi & \text{in } \Omega, \\
Rw = 0 & \text{on } \Gamma, \\
\phi(x_0) = 0 & \text{for an } x_0 \in \overline{\Omega}.
\end{cases}
\]

(4.15)

Applying the orthogonality (4.10), (4.14), approximation property (2.5), and inequalities (4.5) and (4.9), we have

\[
((L'e', \Phi)) = \left| (L'e', L'(w, \phi, \psi) + \sum_{i=1}^{N^*} \alpha_i I_i) \right| \\
= \left| (L'e', L'(w - v^h, \phi - \rho^h, \psi - \gamma^h)) \right| \\
\leq \|L'e'\|_0 \cdot \|L'(w - v^h, \phi - \rho^h, \psi - \gamma^h)\|_0 \\
\leq C\|e'\|_1 \cdot \left( \left\| \begin{array}{c} w - v^h \\ \phi - \rho^h \\ \psi - \gamma^h \end{array} \right\|_1 \\
\leq Ch\|e'\|_1 \cdot (\|w\|_2 + \|\phi\|_2 + \|\psi\|_2) \\
\leq Ch\|e'\|_1 \left| L'(w, \phi, \psi) + \sum_{i=1}^{N^*} \alpha_i I_i \right|_1 \\
= Ch\|e'\|_1 \|\Phi\|_1.
\]

Therefore,

\[
\|L'e'\|_{-1} \leq Ch\|e'\|_1, \quad \text{where } e' = \begin{pmatrix} u - u^h \\ g - g^h \\ k - k^h \end{pmatrix}.
\]

(4.16)

In the special case in which \( k = k^h \equiv 0 \), we have

\[
\|Le\|_{-1} \leq Ch\|e\|_1, \quad \text{where } e = \begin{pmatrix} u - u^h \\ g - g^h \end{pmatrix}.
\]

(4.17)

From (4.8) and (4.9) the next theorem follows.

**Theorem 4.3.** Under the same assumptions as in Theorem 4.1 and Corollary 4.2,

\[
\|e\|_0 \leq Ch^s (\|u\|_s + \|g\|_s), \quad 1 \leq s \leq r + 1,
\]

(4.18)

where \( C > 0 \) is independent of \( h \).

5. Numerical experiments. We have considered the general case of the grad-div linear system (2.1) in the previous paragraphs. In this section we give some numerical examples illustrating the theory developed above. Two kinds of elements will be applied for the unit square, \( 0 < x, y < 1 \). In the first instance, piecewise linear elements with support in directional triangles are used as in Fig. 1(a); the square is subdivided into \( N^2 \) squares of side \( h = \frac{1}{N} \) and then subdivided further, each small
square into two triangles by drawing a diagonal. In the second example the bilinear element is of the form shown in Fig. 1(b).

\[ \text{grad} \ (g) - u = 0 \quad \text{in } \Omega, \]
\[ \text{div} \ u = 2(x - x^2) + 2(y - y^2) \quad \text{in } \Omega, \]
\[ u \times n = 0 \quad \text{on } \Gamma, \]
\[ g(0) = 0. \]

Thus the exact solution of the problem is

\[ \begin{cases} u = ((2x - 1)(y - y^2), (2y - 1)(x - x^2)), \\ g = (x - x^2)(y^2 - y). \end{cases} \]

Table 1 exhibits the numerical results for the piecewise linear elements in directional triangulation and for the bilinear quadrilaterals, respectively. It indicates the size of the vector error \( e = (u - u^h, g - g^h) \) with respect to both \( H^1 \) and \( L^2 \) norms.

<table>
<thead>
<tr>
<th>( h^{-1} )</th>
<th>Number of elements</th>
<th>Directional triangulation</th>
<th>Bilinear quadrilateral</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( h^{-1} \cdot |e|_1 )</td>
<td>( h^{-2} \cdot |e|_0 )</td>
</tr>
<tr>
<td>4</td>
<td>54</td>
<td>1.03</td>
<td>0.401</td>
</tr>
<tr>
<td>6</td>
<td>118</td>
<td>1.06</td>
<td>0.416</td>
</tr>
<tr>
<td>8</td>
<td>206</td>
<td>1.07</td>
<td>0.426</td>
</tr>
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<tr>
<td>24</td>
<td>1774</td>
<td>1.08</td>
<td>0.499</td>
</tr>
</tbody>
</table>

**Fig. 1.** (a) Directional triangle; (b) Bilinear quadrilateral.

**5.1. Numerical example 1.**
5.2. Numerical example 2.

\[
\begin{aligned}
\text{grad } (g) - u &= 0 \quad \text{in } \Omega, \\
\text{div } u &= -2\pi^2 \cos(\pi x) \cdot \cos(\pi y) \quad \text{in } \Omega, \\
u \cdot n &= 0 \quad \text{on } \Gamma, \\
g(0) &= 0.
\end{aligned}
\]

The exact solution of this problem is

\[
\begin{aligned}
u &= (-\pi \sin(\pi x) \cdot \cos(\pi y), -\pi \cos(\pi x) \cdot \sin(\pi y)), \\
g &= \cos(\pi x) \cdot \cos(\pi y).
\end{aligned}
\]

Table 2 gives the size of the vector error for Example 2.

<table>
<thead>
<tr>
<th>$h^{-1}$</th>
<th>Number of elements</th>
<th>Directional triangulation $h^{-1} \cdot |e|_{1}$</th>
<th>$h^{-2} \cdot |e|_{0}$</th>
<th>Bilinear quadrilateral $h^{-1} \cdot |e|_{1}$</th>
<th>$h^{-2} \cdot |e|_{0}$</th>
</tr>
</thead>
<tbody>
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<td>54</td>
<td>15.37</td>
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5.3. Discussion of the examples. Both Examples 1 and 2 support the error estimates given in (4.8) and (4.15), i.e., the error of vector $e = (u - u^h, g - g^h)$ in $H^1$-norm and in $L^2$-norm are proportional to $h$ and $h^2$, respectively. But in each of these two numerical results the constants $h^{-1} \|e\|_{1}$ and $h^{-2} \|e\|_{0}$ differ. The grids for these computations do have different shape; if we choose the grids so that in each case the number of elements is the same, the above constants, which govern the rate of convergence, still differ. This is so even though the CPU times for the respective computations are almost equal. Recall that $\|e\|_0 = (\|u - u^h\|_2^2 + \|g - g^h\|_2)_{1/2}$. Roughly speaking, this quantity may be regarded as measuring the size of the $H^1$ error in $g - g^h$, which error is of order $O(h^2)$.

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REFERENCES


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