Introduction

Although the mathematical formulation of the flexibility and stiffness methods are similar, the physical concepts involved are different.

We found that in the flexibility method, the unknowns were the redundant actions. In the stiffness method the unknown quantities will be the joint displacements. Hence, the number of unknowns is equal to the degree of kinematic indeterminacy for the stiffness method.

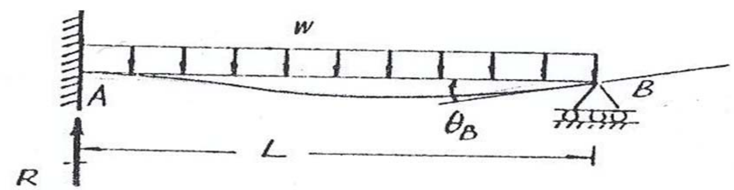
Flexibility Method:

- Unknown redundant actions $\{Q\}$ are identified and structure is released
- Released structure is statically determinate
- Flexibility matrix is formulated and redundant actions $\{Q\}$ are solved for
- Other unknown quantities in the structure are functionally dependent on the redundant actions

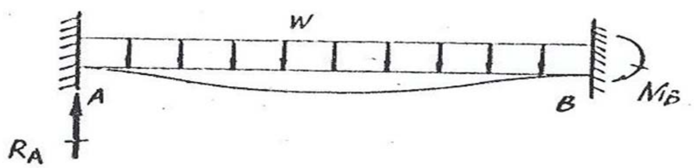
Stiffness Method:

- Unknown joint displacements $\{D\}$ are identified and structure is restrained
- Restrained structure is kinematically determinate, i.e., all displacements are zero
- Stiffness matrix is formulated and unknown joint displacements $\{D\}$ are solved for
- Other unknown quantities in the structure are functionally dependent on the displacements.

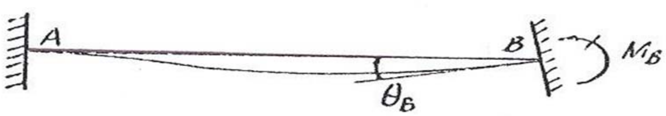
Actual Beam



Restrained Beam #1



Restrained Beam #2



Neglecting axial deformations, the beam to the left is kinematically indeterminate to the first degree. The only unknown is a joint displacement at *B*, that is the rotation. We alter the beam such that it becomes kinematically determinate by making the rotation θ_B zero. This is accomplished by making the end *B* a fixed end. This new beam is then called the restrained structure.

Superposition of restrained beams #1 and #2 yields the actual beam.

Due to the uniform load w, the moment ${}_1M_B$

$$ {}_1M_B = -\frac{wL^2}{12} $$

is developed in restrained beam #1. The moment ${}_1M_B$ is an action in the restrained structure corresponding to the displacement θ_B in the actual beam. The actual beam does not have zero rotation at B. Thus for restrained beam #2 an additional couple at B is developed due to the rotation θ_B. The additional moment is equal in magnitude but opposite in direction to that on the loaded restrained beam.

$$ {}_2M_B = \frac{4EI}{L}\,\theta_B $$

Imposing equilibrium at the joint B in the restrained structure

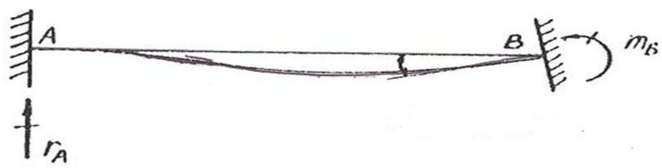
$$ \sum M = -\frac{wL^2}{12} + \frac{4EI}{L}\,\theta_B = 0 $$

yields

$$ \theta_B = \frac{wL^3}{48EI} $$

In a manner analogous to that developed for the flexibility method, we seek a way to consider the previous simple structure under the effect of a unit load. We also wish to utilize the superposition principle. Both will help develop a systematic approach to structures that have a higher degree of kinematic indeterminacy.

The effect of a unit rotation on the previous beam is depicted below



Here the moment applied m_B will produce a unit rotation at B. Since m_B is an action corresponding to the rotation at θ_B and is caused by a unit rotation, then m_B is a stiffness coefficient for the restrained structure. The value of m_B is

$$m_B = \frac{4EI}{L}$$

Again, equilibrium at the joint is imposed. The couple in the restrained beam from the load on the beam will be added to the moment m_B (corresponding to a unit value of θ_B) multiplied by θ_B. The sum of these two terms must give the moment in the actual beam, which is zero, i.e.,

$$M_B + m_B\theta_B = 0$$

or

$$-\frac{wL^2}{12} + \left(\frac{4EI}{L}\right)\theta_B = 0$$

Solving for θ_B yields once again

$$\theta_B = \frac{wL^3}{48EI}$$

The positive sign indicates the rotation is counterclockwise.

This seems a little simple minded, but the systematic approach of applying the principle of superposition will allow us to analyze more complex structures.

Having obtained θ_B then other quantities, such as member end-actions and reactions can be computed. For example, the reaction force R acting at A can be computed by summing the force R_A in the restrained structure due to loads and the force r_A multiplied by θ_B, i.e.,

$$R = R_A + r_A\theta_B$$

The forces R_A and r_A are

$$R_A = \frac{wL}{2} \qquad r_A = \frac{6EI}{L^2}$$

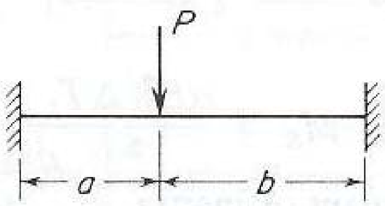
thus

$$R = \frac{wL}{2} + \frac{6EI}{L^2}\left(\frac{wL^3}{48EI}\right) = \frac{5wL}{8}$$

Useful Beam Tables

The next several beam cases will prove useful in establishing components of the stiffness matrix. Consult your Steel Design manual for many others not found here.

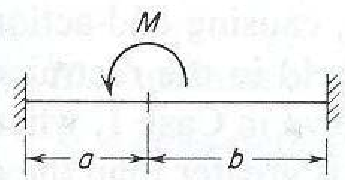
1



$$M_A = \frac{Pab^2}{L^2} \quad M_B = -\frac{Pa^2b}{L^2}$$

$$R_A = \frac{Pb^2}{L^3}(3a + b) \quad R_B = \frac{Pa^2}{L^3}(a + 3b)$$

2

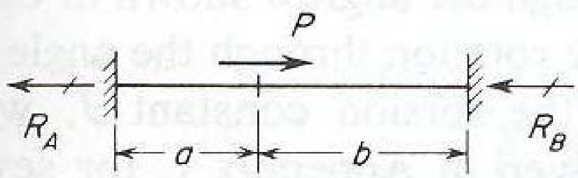


$$M_A = \frac{Mb}{L^2}(2a - b)$$

$$M_B = \frac{Ma}{L^2}(2b - a)$$

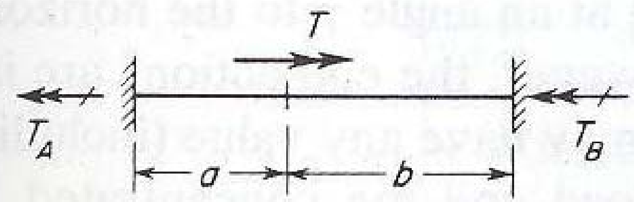
$$R_A = -R_B = \frac{6Mab}{L^3}$$

3



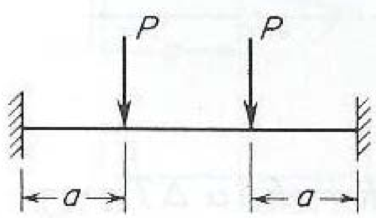
$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

4



$$T_A = \frac{Tb}{L} \qquad T_B = \frac{Ta}{L}$$

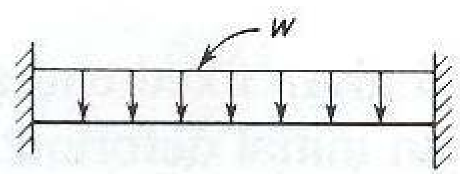
5



$$M_A = -M_B = \frac{Pa}{L}(L - a)$$

$$R_A = R_B = P$$

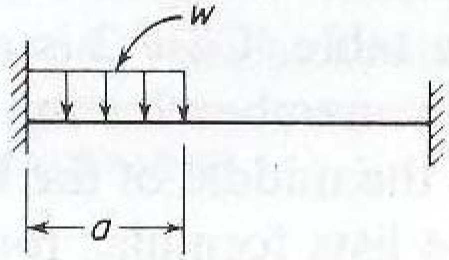
6



$$M_A = -M_B = \frac{wL^2}{12}$$

$$R_A = R_B = \frac{wL}{2}$$

7



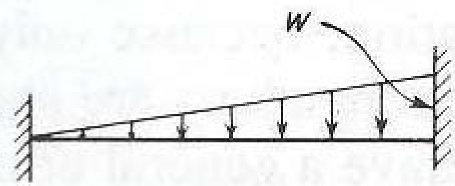
$$M_A = \frac{wa^2}{12L^2}(6L^2 - 8aL + 3a^2)$$

$$M_B = -\frac{wa^3}{12L^2}(4L - 3a)$$

$$R_A = \frac{wa}{2L^3}(2L^3 - 2a^2L + a^3)$$

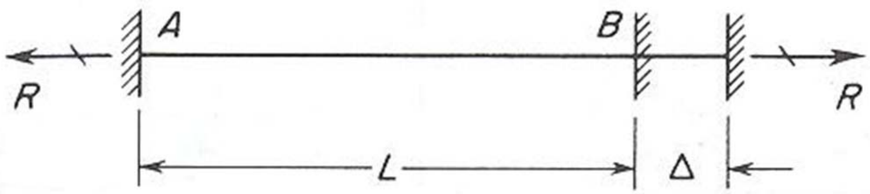
$$R_B = \frac{wa^3}{2L^3}(2L - a)$$

8

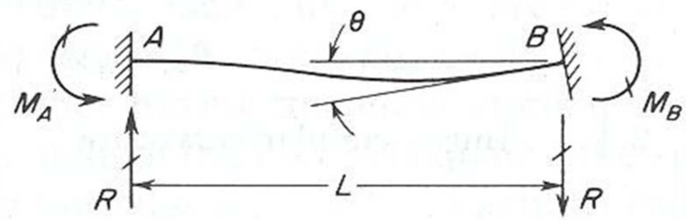


$$M_A = \frac{wL^2}{30} \qquad M_B = -\frac{wL^2}{20}$$

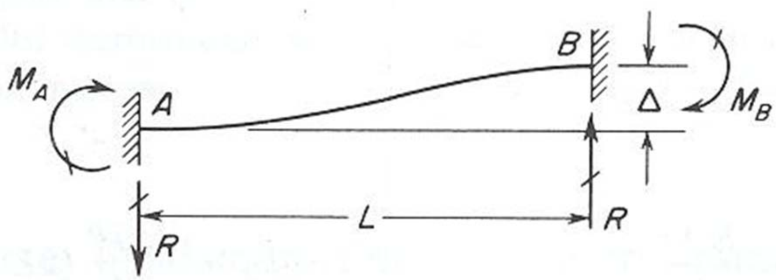
$$R_A = \frac{3wL}{20} \qquad R_B = \frac{7wL}{20}$$



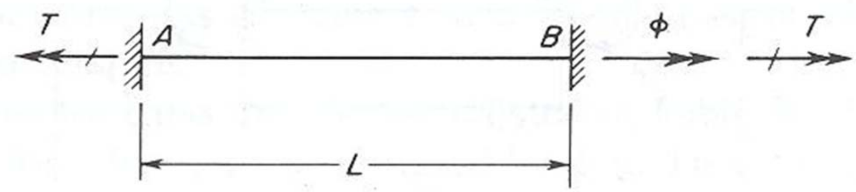
$$R = \frac{EA\Delta}{L}$$



$$M_A = \frac{2EI\theta}{L} \qquad M_B = \frac{4EI\theta}{L} \qquad R = \frac{6EI\theta}{L^2}$$



$$M_A = M_B = \frac{6EI\Delta}{L^2} \qquad R = \frac{12EI\Delta}{L^3}$$



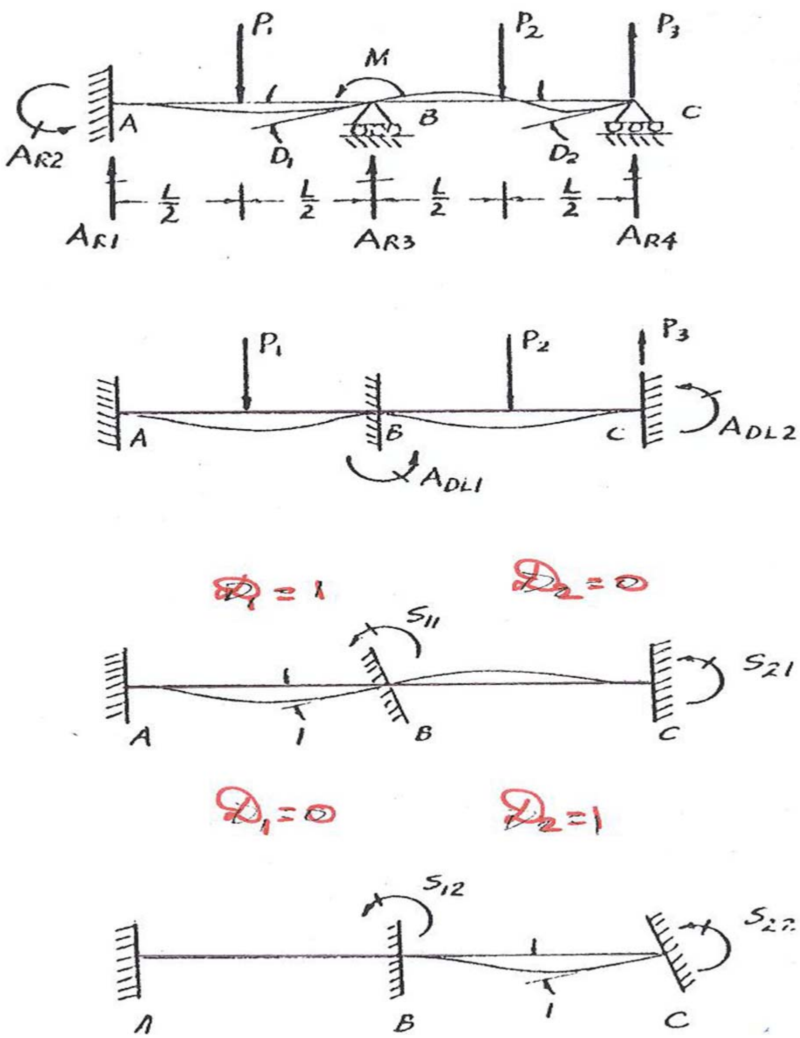
$$T = \frac{GJ\phi}{L}$$

G = shear modulus of elasticity

J = torsion constant

Note that every example cited have fixed-fixed end conditions. All are kinematically determinate.

Multiple Degrees of Kinematic Indeterminacy

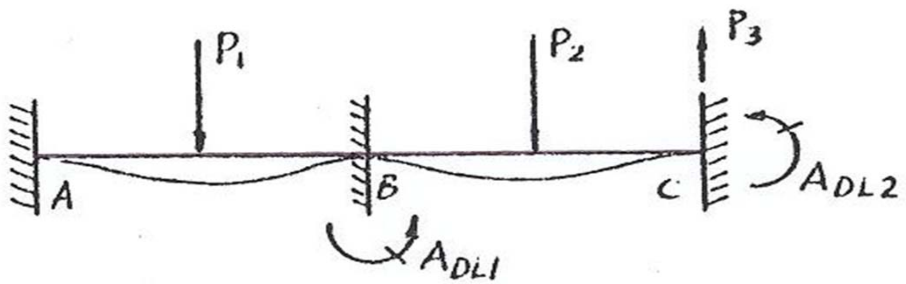


If a structure is kinematically indeterminate to more than one degree a more generalized matrix notation will be utilized.

Consider the beam to the left with a constant flexural rigidity, *EI*. Since rotations can occur at joints *B* and *C*, the structure is kinematically indeterminate to the second degree when axial displacements are neglected.

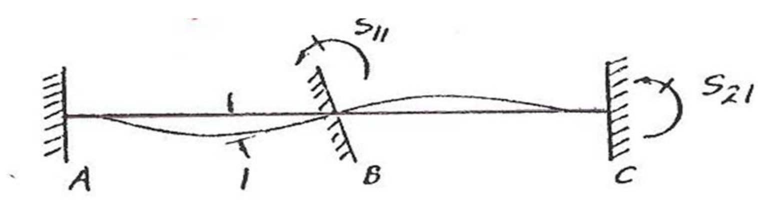
Designate the unknown rotations as D_1 (and the associated bending moment as A_{D1}) and D_2 (with a bending moment A_{D2}). Assume counterclockwise rotations as positive. The unknown displacements are determined by applying the principle of superposition to the bending moments at joints *B* and *C*.

All loads except those corresponding to the unknown joint displacements are assumed to act on the restrained structure. Thus only actions P_1 and P_2 are shown acting on the restrained structure.

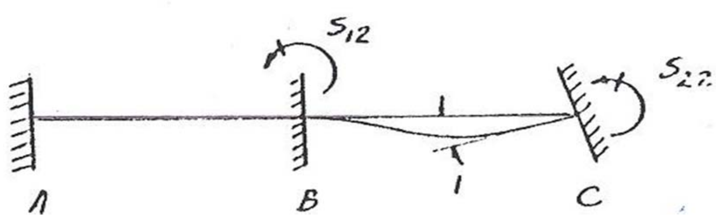


The moments A_{DL1} and A_{DL2} are the actions of the restraints associated with D_1 (A_{D1}) and D_2 (A_{D2}) respectively. The notation in parenthesis will help with the matrix notation momentarily.

In order to generate the stiffness coefficients at joints *B* and *C*, unit values of the unknown displacements D_1 and D_2 are induced in separately restrained structures.



In the restrained beam to the left a unit rotation is applied to joint *B*. Thus the actions induced in the restrained structure corresponding to D_1 and D_2 are the stiffness coefficients S_{11} and S_{21}, respectively



In the restrained beam to the left a unit rotation is applied to joint *B*. Thus the actions induced in this restrained structure corresponding to D_1 and D_2 are the stiffness coefficients S_{12} and S_{22}, respectively

All the stiffness coefficients in the figures have two subscripts (S_{ij}). The first subscript identifies an action associated with an unknown displacement (D_i). The second subscript denotes where the unit displacement is being applied. Stiffness coefficients are taken as positive when the action represented by the coefficient is in the same direction as the ith displacement.

Two superposition equations describing the moment conditions on the original structure may now be expressed at joints *B* and *C*. The superposition equations are

$$A_{D1} = A_{DL1} + S_{11}D_1 + S_{12}D_2$$

$$A_{D2} = A_{DL2} + S_{21}D_1 + S_{22}D_2$$

The two superposition equations express the fact that the actions in the original structure are equal to the corresponding actions in the restrained structure due to the loads plus the corresponding actions in the restrained structure under the unit displacements multiplied by the displacements themselves. These equations can be expressed in matrix format as

$$\{A_D\} = \{A_{DL}\} + [S]\{D\}$$

here

$$\{A_D\} = \begin{Bmatrix} A_{D1} \\ A_{D2} \end{Bmatrix}$$

$$\{A_{DL}\} = \begin{Bmatrix} A_{DL1} \\ A_{DL2} \end{Bmatrix}$$

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

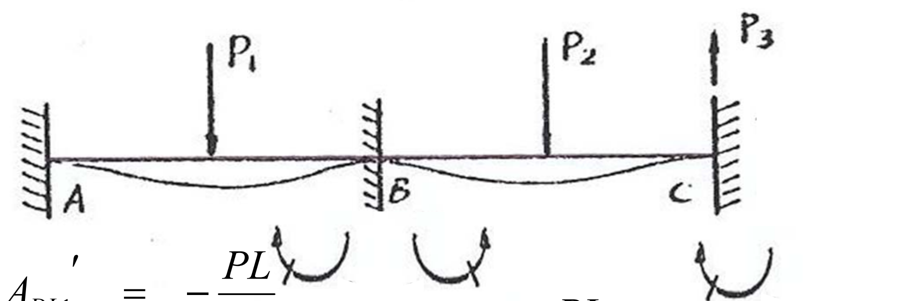
$$\{D\} = \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}$$

and

$$\{D\} = [S]^{-1}\{\{A_D\} - \{A_{DL}\}\}$$

With

$$P_1 = 2P$$
$$M = PL$$
$$P_2 = P$$
$$P_3 = P$$

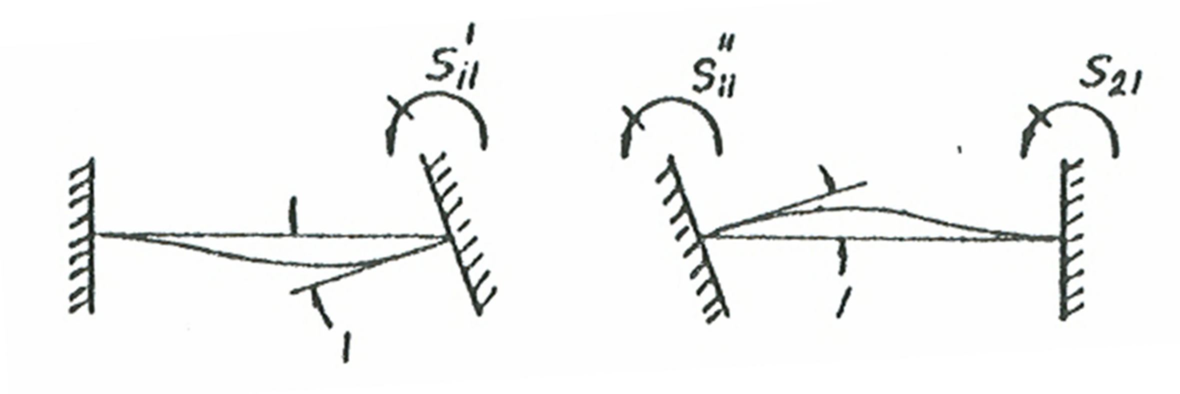


$$A_{DL1}{}' = -\frac{PL}{4} \qquad A_{DL1}{}'' = +\frac{PL}{8} \qquad A_{DL1} = -\frac{PL}{8}$$

then

$$\{A_D\} = \begin{Bmatrix} PL \\ 0 \end{Bmatrix} \qquad \{A_{DL}\} = \begin{Bmatrix} -\frac{PL}{4}+\frac{PL}{8}=-\frac{PL}{8} \\ -\frac{PL}{8} \end{Bmatrix}$$

The next step is the formulation of the stiffness matrix. Consider a unit rotation at *B*



thus

$$S'_{11} = \frac{4EI}{L} \qquad S''_{11} = \frac{4EI}{L}$$

$$S_{11} = S'_{11} + S''_{11} = \frac{8EI}{L}$$

$$S_{21} = \frac{2EI}{L}$$

With a unit rotation at C

$$S_{22} = \frac{4EI}{L} \qquad S_{12} = \frac{2EI}{L}$$

and the stiffness matrix is

$$S = \frac{EI}{L}\begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix}$$

The inverse of the stiffness matrix is

$$[S]^{-1} = \frac{L}{14EI}\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$$

which leads to the following expression

$$\{D\} = \frac{L}{14EI}\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\left\{ \begin{Bmatrix} Pl \\ 0 \end{Bmatrix} - \begin{Bmatrix} -\dfrac{PL}{8} \\ -\dfrac{PL}{8} \end{Bmatrix} \right\}$$

$$= \frac{PL^2}{112EI}\begin{Bmatrix} 17 \\ -5 \end{Bmatrix}$$

Member End-Actions And Reactions

We would now like to develop the matrix equations for determining member end actions and reactions using stiffness methods. The procedure closely follows the procedure developed for the flexibility method. First member end-actions due to the external loads, denoted by $\{A_{ML}\}$, are determined. Then the contributions of the member end-actions caused by unit displacements multiplied by the now known actual displacements are added. Thus

$$\{A_M\} = \{A_{ML}\} + \{A_{MD}\}\{D\}$$

Here:

$\{A_M\}$ is the vector of member end actions on the actual structure

$\{A_{ML}\}$ is the vector of member end actions due to the external loads on the restrained structure.

$\{A_{MD}\}$ is the matrix of member end-actions due to unit values of the displacements on the restrained structure

A similar equation can be written for the reactions, i.e.,

$$\{A_R\} = \{A_{RL}\} + \{A_{RD}\}\{D\}$$

Here:

$\{A_R\}$ is the vector reactions in the actual structure
$\{A_{RL}\}$ is the vector of reactions due to the external loads on the restrained structure
$\{A_{RD}\}$ is the matrix of reactions due to unit values of the displacements on the restrained structure

Example 11.1