Section 7: PRISMATIC BEAMS

Beam Theory

There are two types of beam theory available to craft beam element formulations from. They are

- Bernoulli-Euler beam theory
- Timoshenko beam theory

One learns the details of Bernoulli-Euler beam theory as undergraduates in a Strength of Materials course. Timoshenko beam theory is used when the effects on deformation from shear is significant. The conditions under which shear deformations can not be ignored are spelled out later in the section in the discussion on strain energy.

The derivation for both theories are presented here. Bernoulli-Euler beam theory is discussed first. In the next figure a deformed beam is superimposed on a a beam element represented by a straight line. We represented axial force elements by a line element in a graph that maps deformations in a very similar manner. However, for a beam element we should be concerned with

- Axial translations
- Beam deflections
- Rotations (bending and torsional)
However, in order to develop the two beam theories torsional rotations and axial translations will be ignored for the moment and the focus will be on effects from beam bending. In the figure below the original position of the centroidal axis coincides with the x-axis. The cross sectional area is constant along the length of the beam. A second moment of the cross sectional area with respect to z-axis is identified as $I_z$. The beam is fabricated from a material with a Young’s modulus $E$. Nodes are identified at either end of the beam element.
The key assumption for Bernoulli-Euler beam theory is that plane sections remain plane and also remain perpendicular to the deformed centroidal axis.

The previous figure shows a beam element based on this theory of length, $L$, with transverse end displacements, $v_1$ and $v_2$, rotation of the end planes $\theta_1$ and $\theta_2$ and rotation of the neutral axis, $\mu_1$ and $\mu_2$ at nodes 1 and 2.

The requirement that cross-sections remain perpendicular to the neutral axis means that

$$\begin{align*}
\theta_1 &= \mu_1 \\
\theta_2 &= \mu_2
\end{align*}$$

Note that

$$\mu = \frac{dv}{dx}$$

where the sign convention that clockwise rotations are negative is adopted.
Recall from elementary calculus the curvature of a line at a point may be expressed as

\[
\frac{1}{\rho} = \frac{d^2 y}{d x^2} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}
\]

In the case of an elastic curve rotations are small, thus the derivative \( \frac{dy}{dx} \) is small and squaring a small quantity produces something smaller still, thus

\[
\frac{1}{\rho} \approx \frac{d^2 y}{d x^2}
\]

Given a prismatic beam subjected to pure bending

\[
\frac{1}{\rho} = \frac{M}{EI}
\]

When the beam is subjected to transverse loads this equation is still valid provided St. Venant’s principle is not violated. However the bending moment and thus the curvature will vary from section to section.
Thus

\[ \frac{1}{\rho} = \frac{M(x)}{EI} \]

And

\[ \frac{d^2v}{dx^2} = \frac{M(x)}{EI} \]

In addition

\[
\frac{dv}{dx} = \tan(\theta) = \theta = \frac{1}{EI} \int M(x)dx + C_1
\]

The assumption that plane sections remain perpendicular to the centroidal axis necessarily implies that shear strain, \( \gamma_{xy} \), is zero. This in turn implies that shear stress and shear force are zero. The only load case that results in zero shear force is a constant (uniform) bending moment and thus Bernoulli-Euler theory strictly holds only for this case.
As we will see later Bernoulli-Euler beam theory is acceptable only for long slender beams. Errors incurred in displacements by ignoring shear effects are of the order of \((d/L)^2\), where \(d\) is the depth of a beam and \(L\) is the length.

In the case where a beam is relatively short or deep, shear effects can, however, be significant (Timoshenko, 1957). The critical difference in Timoshenko theory is that the assumption that plane sections remain plane and perpendicular to the neutral axis is relaxed to allow plane sections to undergo a shear strain \(\gamma\). The plane section still remains plane but rotates by an amount, \(\theta\), equal to the rotation of the neutral axis, \(\mu\), minus the shear strain \(\gamma\) as shown in the previous figure. The rotation of the cross section is thus

\[
\theta = \mu - \gamma
\]

\[
= \frac{dv}{dx} - \gamma
\]

Taking derivatives of both sides with respect to \(x\) leads to

\[
\frac{d\theta}{dx} = \frac{d^2v}{dx^2} - \frac{d\gamma}{dx}
\]
Thus for a Timoshenko beam 

\[ \frac{d^2v}{dx^2} - \frac{d\gamma}{dx} = \frac{M(x)}{EI} \]

where for a Bernoulli-Euler beam 

\[ \frac{d^2v}{dx^2} = \frac{M(x)}{EI} \]

The differences will come into play when strain energies are computed using one theory or the other. It turns out that a shear correction factor is introduced to the element stiffness matrix for a Bernoulli-Euler beam element to obtain the element stiffness matrix for a Timoshenko beam element.

The issue of short deep beams can arise more often than one might think. For civil engineers consider a girder supporting a short span in a bridge for a major transportation artery (rail or automotive). Another application would be the use of a short deep beam as a beam seat support.
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Figure 3-16. Cross-frame nodal shears and moments and equivalent Euler-Bernoulli beam shears and moments based on shear analogy.

The horizontal loads are concentrated on the right-hand side, and the truss is anchored on the right-hand side. It should be noted that the vertical members at the sides of the cross frames represent the stiffening of the girders and connection plates, which typically involve a larger effective area than the cross frames themselves. The unit load is applied on the right-hand side and the truss is supported on the left-hand side in Figure 3-15 such that no deformations of the end vertical elements come into play. The equivalent beam cross-frame stiffness is obtained by equating the relative deflection to the Euler-Bernoulli beam solution ($EI/2$).

Figure 3-16 shows the nodal shears and moments for both the physical cross-frame and the equivalent Euler-Bernoulli beam in this problem. That is, the nodal shears and moments are identical for the equivalent beam idealization and the physical truss. In this case, however, it should be noted that a large portion of the vertical displacement in the physical truss is due to shearing type deformations whereas the Euler-Bernoulli beam does not include any consideration of shear deformations. Therefore, the equivalent moment of inertia is assumed “artificially reduced” to account for these large shearing deformations in the shear analogy approach.

3.2.3 Equivalent Beam Stiffness for a Timoshenko Beam Element

Figure 3-17 illustrates the first step of a more accurate approach for the calculation of the cross-frame equivalent beam stiffness. This approach simply involves the calculation of an equivalent moment of inertia, $I_{eq}$, as well as an equivalent shear area, $A_{eq}$, for a shear-deformable (Timoshenko) beam element representation of the cross-frame. In this approach, the equivalent...
In static analyses the differences between the two theories is not pronounced until the aspect ratio defined as

$$AR = \frac{Beam \ length}{Beam \ depth} = \frac{L}{d}$$

or $l/h$ in the figures below, is less than 10. The differences become more pronounced in a vibrations problem.
Preliminaries – Beam Elements

The stiffness matrix for any structure is composed of the sum of the stiffnesses of the elements. Once again, we search for a convenient approach to sum the element stiffnesses in some systematic fashion when the structure is a beam.

Consider the following prismatic beam element:

This element can be from a three-dimensional frame so that all possible forces and moments can be depicted at each node.

The member is fully restrained and it is convenient to adopt an orthogonal axes oriented to the member.
The $x_m$-axis is a centroidal axis, the $x_m$ - $y_m$ plane and the $x_m$ - $z_m$ plane are principle planes of bending. We assume that the shear center and the centroid of the member coincide so that twisting and bending are not coupled.

The properties of the member are defined in a systematic fashion. The length of the member is $L$, the cross sectional area is $A_x$. The principle moments of inertial relative to the $y_m$ and $z_m$ axes are $I_y$ and $I_z$, respectively. The torsional constant $J$ ($= I_y + I_z$ only for a circular cross section, aka, the polar moment of inertia only for this special case) will be designated $I_x$ in the figures that follow.

The member stiffnesses for the twelve possible types of end displacement are summarized in the next section.
In the stiffness method the unknown quantities will be the joint displacements. Hence, the number of unknowns is equal to the degree of kinematic indeterminacy for the stiffness method.

Briefly, the unit load (force or moment) method to determine components of a stiffness matrix proceeds as follows:

- Unknown nodal displacements are identified and structure is restrained
- Restrained structure is kinematically determinate, i.e., all displacements are zero
- Stiffness matrix is formulated by successively applying unit loads at the now restrained unknown nodal displacements

To demonstrate the approach for beams a single span beam is considered. This beam is depicted on the following page. Nodes (joints in Matrix Methods) are positioned at either end of the beam.
Neglecting axial deformations, as well as displacements and rotations in the $z_M$-direction the beam to the left is kinematically indeterminate to the first degree. The only unknown is a joint displacement at $B$, i.e., the rotation. We alter the beam such that it becomes kinematically determinate by making the rotation $\theta_B$ zero. This is accomplished by making the end $B$ a fixed end. This new beam is then called the restrained structure.

Superposition of restrained beams #1 and #2 yields the actual beam.
Due to the uniform load $w$, the moment $M_B$ is

$$1 M_B = \frac{-wL^2}{12}$$

is developed in restrained beam #1. The moment $M_B$ is an action in the restrained structure corresponding to the displacement $\theta_B$ in the actual beam. The actual beam does not have zero rotation at $B$. Thus for restrained beam #2 an additional couple at $B$ is developed due to the rotation $\theta_B$. The additional moment is equal in magnitude but opposite in direction to that on the loaded restrained beam.

$$2 M_B = \frac{4EI}{L} \theta_B$$

Imposing equilibrium at the joint $B$ in the restrained structure

$$\sum M = \frac{-wL^2}{12} + \frac{4EI}{L} \theta_B = 0$$

yields

$$\theta_B = \frac{wL^3}{48EI}$$
There is a way to analyze the previous simple structure through the use of a unit load. The superposition principle can be utilized. Both superposition and the application of a unit load will help develop a systematic approach to analyzing beam elements as well as structures that have much higher degrees of kinematic indeterminacy.

The effect of a unit rotation on the previous beam is depicted below. The moment applied, $m_B$, will produce a unit rotation at $B$.

$$m_B = \frac{4EI}{L}$$

We now somewhat informally define a stiffness coefficient as the action (force or moment) that produces a unit displacement (translation or rotation) at the point of application of the force or moment.
Again, equilibrium is imposed at the right hand node \((B)\). The moment in the restrained beam from the load on the beam will be added to the moment \(m_B\) multiplied by the rotation in the actual beam, \(\theta_B\). The sum of these two terms must give the moment in the actual beam, which is zero, i.e.,

\[
M_B + m_B \theta_B = 0
\]

or

\[
-\frac{wL^2}{12} + \left(\frac{4EI}{L}\right) \theta_B = 0
\]

In finite element analysis the first term above is an equivalent nodal force generated by the distributed load. Solving for \(\theta_B\) yields once again

\[
\theta_B = \frac{wL^3}{48EI}
\]

The positive sign indicates the rotation is counterclockwise. Note that in finite element analysis, \(\theta_B\) is an unknown displacement (rotation) and \(m_B\) is a stiffness coefficient for the beam element matrix. The beam element is defined by nodes \(A\) and \(B\).
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Beam Tables – Useful Cases

The next several beam cases will prove useful in establishing components of the stiffness matrix. Consult the *AISC Steel Design Manual* or Roark’s text for many others not found here.

\[
M_A = \frac{Pab^2}{L^2} \quad M_B = -\frac{Pa^2b}{L^2}
\]

\[
R_A = \frac{Pb^2}{L^3} (3a + b) \quad R_B = \frac{Pa^2}{L^3} (a + 3b)
\]
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3

\[ R_A = \frac{Pb}{L}, \quad R_B = \frac{Pa}{L} \]

5

\[ M_A = -M_B = \frac{Pa}{L} (L - a) \]
\[ R_A = R_B = P \]

6

\[ T_A = \frac{Tb}{L}, \quad T_B = \frac{Ta}{L} \]

\[ M_A = -M_B = \frac{wL^2}{12} \]
\[ R_A = R_B = \frac{wL}{2} \]
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\[ M_A = \frac{wa^2}{12L^2} (6L^2 - 8aL + 3a^2) \]
\[ M_B = -\frac{wa^3}{12L^2} (4L - 3a) \]
\[ R_A = \frac{wa}{2L^3} (2L^3 - 2a^2L + a^3) \]
\[ R_B = \frac{wa^3}{2L^3} (2L - a) \]
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Note that every example cited have fixed-fixed end conditions. All are kinematically determinate beam elements.
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Multiple Degrees of Kinematic Indeterminacy

Consider the beam to the left with a constant flexural rigidity, $EI$. Since rotations can occur at nodes $B$ as well as $C$, and we do not know what they are, the structure is kinematically indeterminate to the second degree. Axial displacements as well as displacements and rotations in the $z_M$-direction are neglected.

Designate the unknown rotations as $D_1$ (and the associated bending moment from the load applied between nodes as $A_{DL1}$) and $D_2$ (and bending moment $A_{DL2}$ from the load between nodes). Assume counterclockwise rotations as positive. The unknown displacements are determined by applying the principle of superposition to the bending moments at joints $B$ and $C$.

Note: $D$ – displacement (translation or rotation)  
       $A$ – action (force or moment)
All loads except those corresponding to the unknown joint displacements are assumed to act on the restrained structure. Thus only actions $P_1$ and $P_2$ are shown acting on the restrained structure.

The moments $A_{DL1}$ and $A_{DL2}$ are the actions at the restrained nodes associated with $D_1$ ($A_{DL1}$ in *Matrix Methods*, $M_B$, the global moment at node $B$ in *Finite Element Analysis*) and $D_2$ ($A_{DL2}$ in *Matrix Methods*, $M_C$, the global moment at node $C$ in *Finite Element Analysis*) respectively. The notation in parenthesis will help with the matrix notation momentarily.

In addition, numerical subscripts are associated with the unknown nodal displacements, i.e., $D_1$ and $D_2$. These displacements occur at nodes $B$ and $C$. Later when this approach is generalized every node will be numbered, not lettered. In addition, the actions in the figure above that occur between nodes are subscripted for convenience and these subscripts bear no relationship to the nodes.
In order to generate individual values of the beam element stiffness matrix, unit values of the unknown displacements $D_1$ and $D_2$ are induced in separately restrained structures. In the restrained beam to the left a unit rotation is applied to joint $B$. The actions (moments) induced in the restrained structure corresponding to $D_1$ and $D_2$ are the stiffness coefficients $S_{11}$ and $S_{21}$, respectively.

In the restrained beam to the left a unit rotation is applied to joint $B$. The actions (moments) induced in this restrained structure corresponding to $D_1$ and $D_2$ are the stiffness coefficients $S_{12}$ and $S_{22}$, respectively.

All the stiffness coefficients in the figures have two subscripts ($S_{ij}$). The first subscript identifies the location of the action induced by the unit displacement. The second subscript denotes where the unit displacement is being applied. **Stiffness coefficients are taken as positive when the action represented by the coefficient is in the same direction as the $i^{th}$ displacement.**
Two superposition equations describing the moment conditions on the original structure may now be expressed at joints $B$ and $C$. The superposition equations are

\[
A_{D1} = A_{DL1} + S_{11}D_1 + S_{12}D_2
\]

\[
A_{D2} = A_{DL2} + S_{21}D_1 + S_{22}D_2
\]

The two superposition equations express the fact that the actions in the original structure (first figure defining the structure and loads) must be equal to the corresponding actions in the restrained structure due to the loads (the second figure) plus the corresponding actions in the restrained structure under the unit displacements (third and fourth figure) multiplied by the unknown displacements.
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These equations can be expressed in matrix format as

\[ \{A_D\} = \{A_{DL}\} + [S]\{D\} \]

where using the loads applied to the structure

\[ \begin{align*} 
\{A_D\} &= \begin{bmatrix} A_{D1} \\ A_{D2} \end{bmatrix} = \begin{bmatrix} M_B = M \\ M_C = 0 \end{bmatrix} \\
\{A_{DL}\} &= \begin{bmatrix} A_{DL1} \\ A_{DL2} \end{bmatrix} = \begin{bmatrix} -\frac{P_1L}{8} + \frac{P_2L}{8} \\ -\frac{P_2L}{8} \end{bmatrix} \\
\{D\} &= \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \\
[S] &= \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}
\end{align*} \]

and we are looking for the unknown displacements.

\[ \{D\} = [S]^{-1} \{\{A_D\} - \{A_{DL}\}\} \]
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If

\[
P_1 = 2P \\
M = PL \\
P_2 = P \\
P_3 = P
\]

then

\[
\{ A_D \} = \begin{cases} \{ PL \} \\
0
\end{cases} \quad \{ A_{DL} \} = \begin{cases} \frac{-PL}{4} + \frac{PL}{8} = -\frac{PL}{8} \\
-\frac{PL}{8}
\end{cases}
\]

The next step is the formulation of the stiffness matrix. Consider a unit rotation at \( B \).
thus

\[ S_{11}' = \frac{4EI}{L}, \quad S_{11}'' = \frac{4EI}{L} \]

\[ S_{11} = S_{11}' + S_{11}'' = \frac{8EI}{L} \]

\[ S_{21} = \frac{2EI}{L} \]

With a unit rotation at \( C \)

\[ S_{22} = \frac{4EI}{L} \quad S_{12} = \frac{2EI}{L} \]

and the stiffness matrix is

\[
S = \frac{EI}{L} \begin{bmatrix}
8 & 2 \\
2 & 4
\end{bmatrix}
\]
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Element Stiffness Matrix for Bernoulli-Euler Beams

The General Case

Unit Displacement

Node j  Node k

Unit Displacement

Unit Displacement

Unit Displacement

Unit Displacement

Unit Displacement

Unit Displacement

Unit Displacement
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Unit Displacement

(7)

(8)

(9)

(10)

(11)

(12)
The most general stiffness matrix for the beam element depicted in the last two overheads is a $12x12$ matrix we will denote $[k]$, the subscript me implies “member.” Each column represents the actions caused by a separate unit displacement. Each row corresponds to a figure in the previous two overheads, i.e., a figure depicting where the unit displacement is applied.

$$ [k] = \begin{bmatrix} \frac{E A_x}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{E A_x}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12 E I_y}{L^3} & 0 & 0 & 0 & -\frac{6 E I_y}{L^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{12 E I_y}{L^3} & 0 & -\frac{6 E I_y}{L^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G I_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{6 E I_z}{L^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6 E I_z}{L^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{12 E I_y}{L^3} & 0 & 0 & 0 & 0 & 0 & \frac{12 E I_y}{L^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6 E I_y}{L^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G I_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6 E I_z}{L^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6 E I_z}{L^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4 E I_y}{L^2} \\ \end{bmatrix} }
The stiffness matrix for beam elements where axial displacements as well as displacements and rotations in the $z_M$-direction are neglected is smaller than the previous stiffness matrix. The previous stiffness matrix represents the general case for a Bernoulli-Euler beam element. What about stiffness coefficients for a Timoshenko beam?

For a beam element where axial displacements as well as displacements and rotations in the $z_M$-direction are neglected there are four types of displacements at a node. The actions associated with these displacements (one translation and one rotation at each node) are shown at the left and are identified as vectors 1 through 4 in the figure. The four significant displacements correspond to a shear force and bending moment at each node.
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The corresponding 4x4 member stiffness matrix is given below. The elements of this matrix are obtained from cases (2), (6), (8) and (12) in previous overheads.

\[
[k] = \begin{bmatrix}
\frac{12EI_Z}{L^3} & \frac{6EI_Z}{L^2} & -\frac{12EI_Z}{L^3} & \frac{6EI_Z}{L^2} \\
\frac{6EI_Z}{L^2} & \frac{4EI_Z}{L} & -\frac{6EI_Z}{L^2} & \frac{2EI_Z}{L} \\
-\frac{12EI_Z}{L^3} & \frac{6EI_Z}{L^2} & -\frac{12EI_Z}{L^3} & \frac{6EI_Z}{L^2} \\
\frac{6EI_Z}{L^2} & \frac{2EI_Z}{L} & -\frac{6EI_Z}{L^2} & \frac{4EI_Z}{L}
\end{bmatrix}
\]

- No torsion
- No axial deformation
- No biaxial bending
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Displacement Function – Beam Element

We can derive the $4\times4$ stiffness matrix by taking a different approach. Recall that when displacement functions were first introduced for springs, the number of unknown coefficients corresponded to the number of degrees of freedom in the element. For a simple Bernoulli beam element there are four degrees of freedom – two at each node of the element. At the beginning and end nodes the beam element can deflect ($v$) and rotate ($\theta$).

Consider the complete cubic displacement function

$$v(x) = a_1x^3 + a_2x^2 + a_3x + a_4$$

With

$$v(0) = v_1 = a_4$$

$$\frac{dv(0)}{dx} = \theta_1 = a_3$$

$$v(L) = v_2 = a_1L^3 + a_2L^2 + a_3L + v_1$$

$$\frac{dv(L)}{dx} = \theta_2 = 3a_1L^2 + 2a_2L + a_3$$
Solving the previous four equations for the unknowns $a_1$, $a_2$, $a_3$, and $a_4$ leads to

\[
\begin{align*}
    a_1 &= \frac{2}{L^3} (v_1 - v_2) + \frac{1}{L^2} (\theta_1 + \theta_2) \\
    a_2 &= -\frac{3}{L^2} (v_1 - v_2) - \frac{1}{L} (2\theta_1 + \theta_2) \\
    a_3 &= \theta_1 \\
    a_4 &= \theta_1
\end{align*}
\]

Thus

\[
v(x) = \left[ \frac{2}{L^3} (v_1 - v_2) + \frac{1}{L^2} (\theta_1 + \theta_2) \right] x^3
\]

\[
+ \left[ -\frac{3}{L^2} (v_1 - v_2) - \frac{1}{L} (2\theta_1 + \theta_2) \right] x^2 + (\theta_1) x + \theta_1
\]

Using shape functions where

\[
v(x) = [N] \{d\} \]
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and

\[
\{d\} = \begin{bmatrix} \nu_1 \\ \theta_1 \\ \nu_2 \\ \theta_2 \end{bmatrix}
\]

then the shape functions become

\[
\begin{align*}
N_1 &= \frac{1}{L^3} \left( 2x^3 - 3x^2 L + L^3 \right) \\
N_2 &= \frac{1}{L^3} \left( x^3 - 2x^2 L^2 + xL^3 \right) \\
N_3 &= \frac{1}{L^3} \left( -2x^3 + 3x^2 L \right) \\
N_4 &= \frac{1}{L^3} \left( x^3 L - x^2 L^2 \right)
\end{align*}
\]
From *Strength of Materials* the moment-displacement and shear displacement relationships are

\[ M(x) = EI \frac{d^2v}{dx^2} \quad V(x) = EI \frac{d^3v}{dx^3} \]

With the following sign convention for nodal forces, shear forces, nodal bending moments and nodal rotations within a beam element

\[ f_{1y} = V = EI \frac{d^3v(0)}{dx^3} = \frac{EI}{L^3} (12v_1 + 6L \theta_1 - 12v_2 + 6L \theta_2) \]
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\[ m_1 = -M = -EI \frac{d^2 v(0)}{dx^2} = \frac{EI}{L^3} \left( 6L v_1 + 4L^2 \theta_1 - 6L v_2 + 2L^2 \theta_2 \right) \]

Minus signs result from the following traditional sign convention used in beam theory.

At node #2

\[ f_{2y} = -V = EI \frac{d^3 v(L)}{dx^3} = \frac{EI}{L^3} \left( -12 v_1 - 6L \theta_1 + 12 v_2 - 6L \theta_2 \right) \]

\[ m_1 = M = EI \frac{d^2 v(L)}{dx^2} = \frac{EI}{L^3} \left( 6L v_1 + 2L^2 \theta_1 - 6L v_2 + 4L^2 \theta_2 \right) \]

These four equations relate actions (forces and moments) at nodes to nodal displacements (translations and rotations).
The four equations can be place into a matrix format as follows

\[
\begin{bmatrix}
  f_{1y} \\
  m_1 \\
  f_{2y} \\
  m_2
\end{bmatrix}
= \frac{EI}{L^3}
\begin{bmatrix}
  12 & 6L & -12 & 6L \\
  6L & 4L^2 & -6L & 2L^2 \\
  -12 & -6L & 12 & -6L \\
  6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\begin{bmatrix}
  v_1 \\
  \theta_1 \\
  v_2 \\
  \theta_2
\end{bmatrix}
\]

where the element stiffness matrix is

\[
[k] = \frac{EI}{L^3}
\begin{bmatrix}
  12 & 6L & -12 & 6L \\
  6L & 4L^2 & -6L & 2L^2 \\
  -12 & -6L & 12 & -6L \\
  6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\]