**Effects Of Temperature, Pre-strain & Support Displacement**

In the previous sections we have only considered loads acting on the structure. We would also like to consider the effects of

- Temperature changes \( D_{QT} \)
- Prestrain of members \( D_{QP} \)

These effects are taken into account by including them in the calculation of displacements (next page) in the released structure in a manner similar to \( D_{QL} \). The effects will produce displacements in the released structure, and the displacements are associated with the redundant actions \( Q \) in the released structure.

The temperature displacements \( D_{QT} \) in the released structure may be due to either uniform changes in temperature or to differential changes in temperature. A differential change in temperature assumes that the top and the bottom of the member changes temperature and thus will undergo a curvature along the axis of the structural component. A uniform change in temperature will increase or decrease the length of the structural component.
Lecture 9: Flexibility Method

When the matrices \( \{D_{QT}\} \) and \( \{D_{QP}\} \) are found they can be added to the matrix \( \{D_{QL}\} \) of displacements due to loads in order to obtain the sum of all displacements in the released structure. By superposition

\[
\{D_Q\} = \{D_{QL}\} + \{D_{QT}\} + \{D_{QP}\} + [F]\{Q\}
\]

As before the superposition equation is solved for the matrix of redundants \( \{Q\} \).

Consider the possibility of known displacements occurring at the restraints (or supports) of the structure. There are two possibilities to consider, depending on whether the restraint displacements corresponds to one of the redundant actions \( \{Q\} \).

If the displacement does correspond to a redundant, its effect can be taken into account by including the displacement in the vector \( \{D_Q\} \).

In a more general situation there will be restraint displacements that do not correspond to any of the selected redundants. In that event, the effects of restraint displacements must be incorporated in the analysis of the released structure in a manner similar to temperature displacements and prestrains. When restraint displacements occur in the released structure a new matrix \( \{D_{QR}\} \) is introduced.
Thus the sum of all matrices representing displacements in the released structure will be denoted by \( \{D_{QS}\} \) and is expressed as follows

\[
\{D_{QS}\} = \{D_{QL}\} + \{D_{QT}\} + \{D_{QP}\} + \{D_{QR}\}
\]

The generalized form of the superposition equation becomes

\[
\{D_Q\} = \{D_{QS}\} + [F]\{Q\}
\]

When this expression is inverted to obtain the redundants we find that

\[
\{Q\} = [F]^{-1} \left\{ \{D_Q\} - \{D_{QS}\} \right\}
\]
Joint Displacements, Member End Actions And Reactions

In the previous sections we have focused on finding redundants using the flexibility method. After redundants were found other actions in the released structure could be found using equations of equilibrium. When all actions in a structure have been determined it is possible to compute displacements by isolating the individual components of a structure and computing displacements from strength of materials expression. Usually in a structural analysis the displacements of the joints are of primary interest.

Instead of following the procedure just outlined we will now introduce a systematic procedure for calculating joint displacements, member end actions and reactions directly into the flexibility method computations.
Consider the two span beam to the left where we will compute the redundants $Q_1$ and $Q_2$ as well as the joint displacements $D_{J1}$ and $D_{J2}$, as well as reactions $A_{R1}$ and $A_{R2}$.

Joint displacement denoted by $\{D_J\}$ can be either a translation or rotation.

Member end-actions $\{A_M\}$ are the couples and forces that act at the ends of a member when that component is isolated from the remainder if the structure. The sign convention for member end actions will be:

- $+$ when up for translations and forces
- $+$ when counterclockwise for rotation and couples

Reactions other than redundants will be denoted $\{A_R\}$. 

Lecture 9: Flexibility Method
The principle of superposition will be used to obtain the joint displacements \([D_j]\) in the actual structure. In order to do this we need to evaluate the displacements in the released structure. In the released structure the displacements associated with the actual joint displacements are designated \(\{D_{jL}\}\). The rotations at joints B \((= D_{j1})\) and C \((= D_{j2})\) are required. Consider the expressions

\[
D_{j1} = D_{jL1} + D_{jQ11}Q_1 + D_{jQ12}Q_2
\]

Here

- \(D_{j1}\) is the displacement desired, in this case the rotation at joint B.
- \(D_{jL1}\) is the displacement at joint B caused by the external loads in the released structure.
- \(D_{jQ11}\) is the displacement at joint B caused by a unit load at joint B corresponding to the redundant \(Q_1\).
- \(D_{jQ12}\) is the displacement at joint B caused by a unit action at joint C corresponding to the redundant \(Q_2\).

A similar expression can be derived for the rotation at C \((= D_{j2})\), i.e.,

\[
D_{j2} = D_{jL2} + D_{jQ21}Q_1 + D_{jQ22}Q_2
\]
The expressions on the previous slide can be expressed in a matrix format as follows

\[
\{D_J\} = \{D_{JL}\} + [D_{JQ}]\{Q\}
\]

where

\[
\{D_J\} = \begin{pmatrix} D_{J1} \\ D_{J2} \end{pmatrix}, \quad \{D_{JL}\} = \begin{pmatrix} D_{JL1} \\ D_{JL2} \end{pmatrix}, \quad \{Q\} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}
\]

\[
[D_{JQ}] = \begin{bmatrix} D_{JQ11} & D_{JQ12} \\ D_{JQ21} & D_{JQ22} \end{bmatrix}
\]
In a similar manner we can find member end actions via superposition

\[
A_{M1} = A_{ML1} + A_{MQ11}Q_1 + A_{MQ12}Q_2
\]

\[
A_{M2} = A_{ML2} + A_{MQ21}Q_1 + A_{MQ22}Q_2
\]

\[
A_{M3} = A_{ML3} + A_{MQ31}Q_1 + A_{MQ32}Q_2
\]

\[
A_{M4} = A_{ML4} + A_{MQ41}Q_1 + A_{MQ42}Q_2
\]

For the first expression

\[
A_{M1} = \text{is the shear force at } B \text{ on member } AB
\]

\[
A_{ML1} = \text{is the shear force at } B \text{ on member } AB \text{ caused by the external loads on the released structure}
\]

\[
A_{MQ11} = \text{is the shear force at } B \text{ on member } AB \text{ caused by a unit load corresponding to the redundant } Q_1
\]

\[
A_{MQ12} = \text{is the shear force at } B \text{ on member } AB \text{ caused by a unit load corresponding to the redundant } Q_2
\]

The other expressions follow in a similar manner.
Lecture 9: Flexibility Method

The expressions on the previous slide can be expressed in a matrix format as follows

\[
\begin{align*}
\{A_M\} &= \{A_{ML}\} + \{A_{MQ}\}\{Q\}
\end{align*}
\]

where

\[
\begin{align*}
\{A_M\} &= \begin{bmatrix} A_{M1} \\ A_{M2} \\ A_{M3} \\ A_{M4} \end{bmatrix}, \\
\{A_{ML}\} &= \begin{bmatrix} A_{ML1} \\ A_{ML2} \\ A_{ML3} \\ A_{ML4} \end{bmatrix}, \\
\{Q\} &= \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}
\end{align*}
\]

\[
\{A_{MQ}\} = \begin{bmatrix} A_{MQ11} & A_{MQ12} \\ A_{MQ21} & A_{MQ22} \\ A_{MQ31} & A_{MQ32} \\ A_{MQ41} & A_{MQ42} \end{bmatrix}
\]
In a similar manner we can find reactions via superposition

\[ A_{R1} = A_{RL1} + A_{RQ11}Q_1 + A_{RQ12}Q_2 \]

\[ A_{R2} = A_{RL2} + A_{RQ21}Q_1 + A_{RQ22}Q_2 \]

For the first expression

- \( A_{R1} \) is the reaction in the actual beam at A
- \( A_{RL1} \) is the reaction in the released structure due to the external loads
- \( A_{RQ11} \) is the reaction at A in the released structure due to the unit action corresponding to the redundant \( Q_1 \)
- \( A_{RQ12} \) is the reaction at A in the released structure due to the unit action corresponding to the redundant \( Q_2 \)

The other expression follows in a similar manner.
The expressions on the previous slide can be expressed in a matrix format as

\[
\{A_R\} = \{A_{RL}\} + [A_{RQ}]\{Q\}
\]

where

\[
\begin{align*}
\{A_R\} &= \begin{bmatrix} A_{R1} \\ A_{R2} \end{bmatrix} \\
\{A_{RL}\} &= \begin{bmatrix} A_{RL1} \\ A_{RL2} \end{bmatrix} \\
\{Q\} &= \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}
\end{align*}
\]

\[
[A_{RQ}] = \begin{bmatrix}
A_{RQ11} & A_{RQ12} \\
A_{RQ21} & A_{RQ22}
\end{bmatrix}
\]
Lecture 9: Flexibility Method

When the effects of joint displacements, member end actions and reactions are accounted for the equation of superposition becomes

\[
\{D_J\} = \{D_{JS}\} + [D_{JQ}]{Q}
\]

here

\[
\{D_{JS}\} = \{D_{JL}\} + \{D_{JT}\} + \{D_{JP}\} + \{D_{JR}\}
\]

where

\[
\{D_{JT}\} = \text{joint displacement due to temperature}
\]

\[
\{D_{JP}\} = \text{joint displacement due to prestrain}
\]

\[
\{D_{JR}\} = \text{joint displacement corresponding to redundants}
\]

Hence there is no need to generalize the expression for \(\{A_M\}\) and \(\{A_R\}\) to account for temperature effects, prestrain and displacement effects. None of these effects will produce any actions or reactions in a statically determinate released structure. Instead the released structure will merely change its configurations to accommodate these effects. The effects of these influences are merely propagated into matrices \(\{A_M\}\) and \(\{A_R\}\) through the value of the redundants \(Q\).
Lecture 9: Flexibility Method

Example

Consider the two span beam to the left where it is assumed that the objective is to calculate the various joint displacements $D_J$, member end actions $A_M$, and end reactions $A_R$. The beam has a constant flexural rigidity $EI$ and is acted upon by the following loads

\begin{align*}
P_1 &= 2P \\
M &= PL \\
P_2 &= P \\
P_3 &= P
\end{align*}
Consider the released structure and the attending moment area diagrams.

The $(M/EI)$ diagram was drawn by parts. Each action and its attending diagram is presented one at a time in the figure starting with actions on the far right.
From first moment area theorem

\[
D_{JL1} = \frac{1}{2} \frac{PL}{EI} (2 + 1) L - \frac{1}{2} \frac{PL}{EI} (1.5 + 0.5) L + \frac{PL}{EI} L - \frac{1}{2} \frac{PL}{EI} \frac{L}{2} \\
= \frac{5 PL^2}{4 EI}
\]

\[
D_{JL2} = \frac{1}{2} \left( \frac{2 PL}{EI} \right) 2 L - \frac{1}{2} \left( \frac{3 PL}{2 EI} \right) \frac{3L}{2} + \left( \frac{PL}{EI} \right) L - \frac{1}{2} \frac{PL}{EI} \frac{L}{2} \\
= \frac{13 PL^2}{8 EI}
\]

\[
[D_{JL}] = \frac{PL^2}{8 EI} \begin{bmatrix} 10 \\ 13 \end{bmatrix}
\]
Lecture 9: Flexibility Method

Using the following free body diagram of the released structure

![Free Body Diagram](image)

Then from the equations of equilibrium

\[
\sum M_A = 0
\]

\[
= A_{RL2} - 2P \frac{L}{2} + PL - P \frac{3L}{2} + P 2L
\]

\[
A_{RL2} = -\frac{PL}{2}
\]

\[
\sum F_y = 0
\]

\[
= A_{RL1} - 2P - P + P
\]

\[
A_{RL1} = 2P
\]
Using a free body diagram from segment AB of the entire beam, i.e.,

\[ \sum F_y = 0 \]
\[ = A_{ML_1} - 2P + 2P \]
\[ A_{ML_1} = 0 \]

\[ \sum M_B = 0 \]
\[ = A_{ML_2} + 2P \frac{L}{2} - \frac{PL}{2} - 2PL \]
\[ A_{ML_2} = \frac{3PL}{2} \]
Using a free body diagram from segment $BC$ of the entire beam, i.e.,

$$\sum F_Y = 0$$

$$= A_{ML3} - P + P$$

$$A_{ML3} = 0$$

$$\sum M_B = 0$$

$$= A_{ML4} - \frac{PL}{2} + PL$$

$$A_{ML4} = -\frac{PL}{2}$$
Thus the vectors $A_{ML}$ and $A_{RL}$ are as follows:

$$A_{ML} = \begin{bmatrix} 0 \\ \frac{3PL}{2} \\ 0 \\ -\frac{PL}{2} \end{bmatrix}$$

$$A_{RL} = \begin{bmatrix} 2P \\ -\frac{PL}{2} \end{bmatrix}$$
Consider the released beam with a unit load at point B

\[
D_{JQ \, 11} = \frac{1}{2} \frac{L}{EI} \frac{L}{L^2} = \frac{L^2}{2 \, EI}
\]

\[
D_{JQ \, 21} = \frac{1}{2} \frac{L}{EI} \frac{L}{L^2} = \frac{L^2}{2 \, EI}
\]
Consider the released beam with a unit load at point $C$

\[ D_{JQ_{12}} = \frac{1}{2} \frac{L}{EI} (2 + 1) L \]

\[ = \frac{3 L^2}{2 EI} \]

\[ D_{JQ_{22}} = \frac{1}{2} \frac{2L}{EI} 2L \]

\[ = \frac{2 L^2}{EI} \]
Thus

\[
\{D_{JQ}\} = \frac{L^2}{2EI} \begin{bmatrix}
1 & 3 \\
1 & 4
\end{bmatrix}
\]

In a similar fashion, applying a unit load associated with \(Q_1\) and \(Q_2\) in the previous cantilever beam, we obtain the following matrices

\[
\{A_{MQ}\} = \begin{bmatrix}
1 & 1 \\
0 & L \\
0 & -1 \\
0 & -L
\end{bmatrix}
\]

\[
[A_{RQ}] = \begin{bmatrix}
-1 & -1 \\
-L & -2L
\end{bmatrix}
\]
Previously (Lecture 5)

\[
\{Q\} = \frac{P}{56} \begin{bmatrix} 60 \\ -64 \end{bmatrix}
\]

with

\[
\{D_j\} = \{D_{jL}\} + [D_{jQ}]\{Q\}
\]

then

\[
\{D_j\} = \frac{PL^2}{112EI} \begin{bmatrix} 17 \\ -5 \end{bmatrix}
\]
Similarly, with

\[ \{A_M\} = \{A_{ML}\} + [A_{MQ}]\{Q\} \]

and knowing \([A_{ML}], [A_{MQ}]\) and \([Q]\) leads to

\[ \{A_M\} = \frac{P}{56} \begin{bmatrix} 5 \\ 20 & L \\ 64 \\ 36 & L \end{bmatrix} \]
Finally with

\[ \{ A_R \} = \{ A_{RL} \} + [A_{RQ}]\{Q\} \]

then knowing \([A_{RL}], [A_{RQ}]\) and \([Q]\) leads to

\[ \{ A_R \} = \frac{P}{56} \begin{bmatrix} 107 \\ 31 \end{bmatrix} \]
Summary Of Flexibility Method

The analysis of a structure by the flexibility method may be described by the following steps:

1. Problem statement
2. Selection of released structure
3. Analysis of released structure under loads
4. Analysis of released structure for other causes
5. Analysis of released structure for unit values of redundant
6. Determination of redundants through the superposition equations, i.e.,

\[
\{D_Q\} = \{D_{QS}\} + [F]\{Q\}
\]

\[
\{D_{QS}\} = \{D_{QL}\} + \{D_{QT}\} + \{D_{QP}\} + \{D_{QR}\}
\]

\[
\{Q\} = [F]^{-1} \left\{\{D_Q\} - \{D_{QS}\}\right\}
\]
Lecture 9: Flexibility Method

7. Determine the other displacements and actions. The following are the four flexibility matrix equations for calculating redundants member end actions, reactions and joint displacements

\[ \{D_J\} = \{D_{JS}\} + [D_{JO}]\{Q\} \]

\[ \{A_M\} = \{A_{ML}\} + [A_{MQ}]\{Q\} \]

\[ \{A_R\} = \{A_{RL}\} + [A_{RQ}]\{Q\} \]

where for the released structure

\[ \{D_{JS}\} = \{D_{JL}\} + \{D_{JT}\} + \{D_{JP}\} + \{D_{JR}\} \]

All matrices used in the flexibility method are summarized in the following tables
**Lecture 9: Flexibility Method**

<table>
<thead>
<tr>
<th>MATRIX</th>
<th>ORDER</th>
<th>DEFINITION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q</td>
<td>q x 1</td>
<td>Unknown redundant actions (q = Number of redundant)</td>
</tr>
<tr>
<td>$D_Q$</td>
<td>q x 1</td>
<td>Displacements in the actual structure Corresponding to the redundant</td>
</tr>
<tr>
<td>$D_{QL}$</td>
<td>q x 1</td>
<td>Displacements in the released structure corresponding to the redundants and due to loads</td>
</tr>
<tr>
<td>$F$ or $D_{QQ}$</td>
<td>q x q</td>
<td>Displacements in the released structure corresponding to the redundants and due unit values of the redundants (Flexibility coefficients)</td>
</tr>
<tr>
<td>$D_{QT}, D_{QP}, D_{QR}$</td>
<td>q x 1</td>
<td>Displacements in the released structure corresponding to the redundants and due to temperature, prestrain, and restraint displacements (other than those in DQ)</td>
</tr>
<tr>
<td>$D_{QS}$</td>
<td>q x 1</td>
<td>$D_{QS} = D_{QL} + D_{QT} + D_{QP} + D_{QR}$</td>
</tr>
</tbody>
</table>
### Lecture 9: Flexibility Method

<table>
<thead>
<tr>
<th>MATRIX</th>
<th>ORDER</th>
<th>DEFINITION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_J$</td>
<td>$j \times 1$</td>
<td>Joint displacement in the actual structure ($j =$ number of joint displacement)</td>
</tr>
<tr>
<td>$D_{JL}$</td>
<td>$j \times 1$</td>
<td>Joint displacements in the released structure due to loads</td>
</tr>
<tr>
<td>$D_{QL}$</td>
<td>$j \times 1$</td>
<td>Joint displacements in the released structure due to unit values of the redundants</td>
</tr>
<tr>
<td>$D_{JT}, D_{JP}, D_{JR}$</td>
<td>$j \times 1$</td>
<td>Joint displacements in the released structure due to temperature, prestrain, and restraint displacements (other than those in $D_Q$)</td>
</tr>
<tr>
<td>$D_{JS}$</td>
<td>$j \times 1$</td>
<td>$D_{JS} = D_{JL} + D_{JT} + D_{JP} + D_{JR}$</td>
</tr>
</tbody>
</table>
Matrix Order Definition

<table>
<thead>
<tr>
<th>MATRIX</th>
<th>ORDER</th>
<th>DEFINITION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_M$</td>
<td>$m \times 1$</td>
<td>Member end actions in the actual structure $(m = \text{Number of end-actions})$</td>
</tr>
<tr>
<td>$A_{ML}$</td>
<td>$m \times 1$</td>
<td>Member end actions in the released structure due to loads</td>
</tr>
<tr>
<td>$A_{MQ}$</td>
<td>$m \times q$</td>
<td>Member end actions in the released structure due to unit values of the redundants</td>
</tr>
<tr>
<td>$A_R$</td>
<td>$r \times 1$</td>
<td>Reactions in the actual structure $(r = \text{number of reactions})$</td>
</tr>
<tr>
<td>$A_{RL}$</td>
<td>$r \times 1$</td>
<td>Reactions in the released structure due to loads</td>
</tr>
<tr>
<td>$A_{RQ}$</td>
<td>$r \times q$</td>
<td>Reactions in the released structure due to unit values of the redundants</td>
</tr>
</tbody>
</table>