Section 4: TRUSS ELEMENTS, LOCAL & GLOBAL COORDINATES

Introduction

The principles for the direct stiffness method are now in place. In this section of notes we will derive the stiffness matrix, both local and global, for a truss element using the direct stiffness method. Here a local coordinate system will be utilized initially and the element stiffness matrix will be transformed into a global coordinate system that is convenient for the overall structure.

Inclined or skewed supports will be discussed.

The principle of minimum potential energy will be utilized to re-derive the stiffness matrices.
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The truss element shown below is assumed to have a constant cross sectional area (A), a modulus of elasticity (E), and an initial length (L). The nodal degrees of freedom are the axial displacements directed along the length of the truss element. In addition the following assumptions are made:

1. The truss element cannot sustain shear forces
2. Any effect of transverse displacements are ignored
3. Hooke’s law applies
Assume

\[ \hat{u} = a_1 + a_2 \hat{x} \]

This is an approximation function. We are assuming through the use of this function that displacements are distributed in an approximately linear fashion along the element.

As noted earlier the total number of unknown coefficients must equal the degrees of freedom associated with the element.
To establish the unknown coefficients consider the boundary conditions of the truss element

\[ \hat{u}(0) = \hat{d}_{1x} = a_1 \quad \hat{u}(L) = \hat{d}_{2x} = \hat{d}_{1x} + a_2 L \]

Thus

\[ a_2 = \left( \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} \right) \]

and in a matrix format

\[ \hat{u} = \hat{d}_{1x} + \left( \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} \right) \hat{x} = N_1 \hat{d}_{1x} + N_2 \hat{d}_{2x} = \begin{bmatrix} 1 - \frac{\hat{x}}{L} & \frac{\hat{x}}{L} \end{bmatrix} \begin{bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{bmatrix} \]
Here

\[ N_1 = 1 - \frac{x}{L} \quad \quad N_2 = \frac{x}{L} \]

are shape functions. Consider the following guidelines as they relate to one dimensional structural elements when selecting a displacement function.

1. Common approximation functions are usually polynomials where the function can be expressed in terms of shape functions.

2. The approximation functions must be continuous through the element.

3. The approximation functions must provide inter-element continuity for all degrees of freedom at each node of every element.

4. The approximation functions must allow for rigid body motion, because rigid body motion can occur in a structure.

With

\[ \dot{u} = a_1 + a_2 \dot{x} \]
The strain displacement relationship can now be cast as

\[ \varepsilon(\hat{x}) = \frac{d\hat{u}}{d\hat{x}} = \frac{d}{d\hat{x}} \left[ \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{bmatrix} \right] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{bmatrix} = \{B\} \begin{bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{bmatrix} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} \]

which is constant along the length of the truss element.
The stress-strain relationship becomes

\[
\sigma(\hat{x}) = E \varepsilon(\hat{x}) = E \{B\} \begin{bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{bmatrix} = E \left( \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} \right)
\]

From elementary mechanics

\[
T = A \sigma = AE \left( \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} \right)
\]
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The nodal forces become

\[ \hat{f}_{1x} = -T \quad \hat{f}_{2x} = T \]

thus

\[ T = -\hat{f}_{1x} = \frac{AE}{L} \left( \hat{d}_{2x} - \hat{d}_{1x} \right) \]

or

\[ T = \hat{f}_{2x} = \frac{AE}{L} \left( \hat{d}_{2x} - \hat{d}_{1x} \right) \]

In a matrix format

\[
\begin{pmatrix}
\hat{f}_{1x} \\
\hat{f}_{2x}
\end{pmatrix}
= \frac{AE}{L}
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\begin{pmatrix}
\hat{d}_{1x} \\
\hat{d}_{2x}
\end{pmatrix}
\]
Consider the situation where

\[
\hat{d}_{1x} = \hat{d}_{2x}
\]

The case where both nodes move the same amount in the same direction. This is definition of rigid body motion. Here

\[
\hat{u} = a_1 + a_2 \hat{x}
\]

\[
= \hat{d}_{1x} + \left( \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} \right) \hat{x}
\]

\[
= \hat{d}_{1x} + \left( \frac{0}{L} \right) \hat{x}
\]

\[
= \hat{d}_{1x}
\]

\[
= a_1
\]
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With

\[ \hat{u} = N_1 \hat{d}_{1x} + N_2 \hat{d}_{2x} \]
\[ = N_1 a_1 + N_2 a_1 \]
\[ = (N_1 + N_2)a_1 \]

this infers that

\[ \hat{u} = a_1 \]
\[ = (N_1 + N_2)a_1 \]

Thus for rigid body motion

\[ 1 = N_1 + N_2 \]
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In addition

\[ \varepsilon(\hat{x}) = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} \]
\[ \sigma(\hat{x}) = E\left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\right) \]
\[ T = AE\left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\right) \]
\[ = \frac{\hat{d}_{2x} - \hat{d}_{2x}}{L} \]
\[ = E\left(\frac{0}{L}\right) \]
\[ = AE\left(\frac{0}{L}\right) \]

Stress, strain and the axial force in the truss element will all be zero for rigid body motion.
The general relationship from the previous page holds for an truss element oriented along the x axis. Thus

\[
\begin{bmatrix}
\hat{k}
\end{bmatrix} = \frac{AE}{L} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\]

is the local stiffness matrix. We note that the local stiffness matrix is symmetric, i.e.,

\[
\hat{k}_{ij} = \hat{k}_{ji}
\]

and square. The global stiffness matrix and global force matrix are assembled using nodal force equilibrium equations, force deformation equations and compatibility equations. Examples of assembling these equations will be given.

\[
[K] = \sum_{e=1}^{N} [k]^{(e)} \quad \{F\} = \sum_{e=1}^{N} \{f\}^{(e)}
\]
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One can quickly populate the global stiffness matrix for a truss structure using the methodology developed for the spring element. Consider the $i^{th}$ truss element:

$i^{th}$ Element stiffness properties transfers into the global stiffness matrix as follows:

$$
\begin{bmatrix}
0 & \vdots & \vdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \vdots & \ddots & \vdots \\
0 & \vdots & \vdots & 0
\end{bmatrix}
$$
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In class example
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Transformation of Vectors

In nearly all finite element analyses it is a necessity to introduce both local (to the element) and global (to the component) coordinate axes.

![Diagram showing transformation of vectors with local and global axes.](image-url)
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In general for forces, the transformation is as follows:

\[
\begin{bmatrix}
F_x \\
F_y
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
f_u \\
f_v
\end{bmatrix}
\]

which appears as follows when applied to a truss element:

\[
\begin{bmatrix}
f_{1x} \\
f_{1y} \\
f_{2x} \\
f_{2y}
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & \cos \theta & -\sin \theta \\
0 & 0 & \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
f_{1x} \\
f_{1y} \\
f_{2x} \\
f_{2y}
\end{bmatrix}
\]

\[
\{f\} = \{\hat{f}\}
\]
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For a true truss element, an element that will not sustain shear forces at the end, then

\[ \hat{f}_{2y} = 0 \]

and

\[
\begin{pmatrix}
\hat{f}_{1y} \\
\hat{f}_{1x} \\
\hat{f}_{2x} \\
\hat{f}_{2y}
\end{pmatrix} =
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & \cos \theta & -\sin \theta \\
0 & 0 & \sin \theta & \cos \theta
\end{bmatrix}
\begin{pmatrix}
f_{1x} \\
f_{1y} \\
f_{2x} \\
f_{2y}
\end{pmatrix}
\]
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This leads to the following system of equations

\[
\begin{align*}
    f_{1x} &= \cos \theta \hat{f}_{1x} - 0 + 0 + 0 \\
    f_{1y} &= \sin \theta \hat{f}_{1x} + 0 + 0 + 0 \\
    f_{2x} &= 0 + 0 + \cos \theta \hat{f}_{2x} - 0 \\
    f_{2y} &= 0 + 0 + \sin \theta \hat{f}_{2x} + 0
\end{align*}
\]

solution leads to (show for homework)

\[
\begin{pmatrix}
    \hat{f}_{1x} \\
    \hat{f}_{2x}
\end{pmatrix}
= 
\begin{bmatrix}
    \cos \theta & \sin \theta & 0 & 0 \\
    0 & 0 & \cos \theta & \sin \theta
\end{bmatrix}
\begin{pmatrix}
    f_{1x} \\
    f_{1y} \\
    f_{2x} \\
    f_{2y}
\end{pmatrix}
\]
A similar relationship holds for nodal displacements between local and global coordinate systems.

\[
\begin{bmatrix}
    \hat{d}_{1x} \\
    \hat{d}_{1y} \\
    \hat{d}_{2x} \\
    \hat{d}_{2y}
\end{bmatrix} =
\begin{bmatrix}
    \cos \theta & -\sin \theta & 0 & 0 \\
    \sin \theta & \cos \theta & 0 & 0 \\
    0 & 0 & \cos \theta & -\sin \theta \\
    0 & 0 & \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
    d_{1x} \\
    d_{1y} \\
    d_{2x} \\
    d_{2y}
\end{bmatrix}
\]

\[
\{d\} = [T^*] \{\hat{d}\}
\]
Inverting this last matrix expression yields (show for homework):

\[
\begin{bmatrix}
\hat{d}_{1x} \\
\hat{d}_{1y} \\
\hat{d}_{2x} \\
\hat{d}_{2y}
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & \sin \theta & 0 & 0 \\
-\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta \\
0 & 0 & -\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\hat{d}_{1x} \\
\hat{d}_{1y} \\
\hat{d}_{2x} \\
\hat{d}_{2y}
\end{bmatrix}
\]

or

\[
\{ \hat{d} \} = [T] \{ d \}
\]

where it is obvious that:

\[
[T] = [T^*]^{-1}
\]

Similarly

\[
\{ \hat{f} \} = [T] \{ f \}
\]

(9)
The element stiffness matrix can now be formulated in terms of the global coordinate system as follows.

In local coordinates

\[
\begin{bmatrix}
\hat{f}_{1x} \\
\hat{f}_{1y} \\
\hat{f}_{2x} \\
\hat{f}_{2y}
\end{bmatrix}
= \frac{EA}{L}
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{d}_{1x} \\
\hat{d}_{1y} \\
\hat{d}_{2x} \\
\hat{d}_{2y}
\end{bmatrix}
\]

With

\[
\begin{align*}
\{\hat{d}\} &= [T] \{d\} \\
\{\hat{f}\} &= [T] \{f\}
\end{align*}
\]

Element nodal forces and displacements in local coordinates

\[
\{\hat{f}\} = [\hat{k}] \{\hat{d}\}
\]

Element stiffness matrix in local coordinates (extended)
then

\[
\{ \hat{f} \} = [ \hat{k} ] \{ \hat{d} \}
\]

\[
[T] \{ f \} = [ \hat{k} ][T] \{ d \}
\]

\[
\{ f \} = [T]^{-1} [ \hat{k} ][T] \{ d \}
\]

\[
\{ f \} = [ k ] \{ d \}
\]

where the element stiffness matrix in terms of global coordinates is

\[
[k] = [T]^{-1} [ \hat{k} ][T]
\]

in a matrix format:

\[
[k] = [T]^{-1} [ \hat{k} ][T] = \frac{EA}{L} \begin{bmatrix}
\cos^2 \theta & \cos \theta \sin \theta & -\cos^2 \theta & -\cos \theta \sin \theta \\
\cos \theta \sin \theta & \sin^2 \theta & -\cos \theta \sin \theta & -\sin^2 \theta \\
-\cos^2 \theta & -\cos \theta \sin \theta & \cos^2 \theta & \cos \theta \sin \theta \\
-\cos \theta \sin \theta & -\sin^2 \theta & \cos \theta \sin \theta & \sin^2 \theta
\end{bmatrix}
\]
In order to load up the component stiffness matrix consider the following simple truss

\[
[k]_1 = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
[k]_2 = \frac{EA}{L} \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}
\]

\[
[k]_3 = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
\]
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Create component stiffness matrix from element stiffness matrices

\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
f_{1x} \\
f_{1y} \\
f_{3x} \\
f_{3y}
\end{bmatrix} = \frac{EA}{L}
\begin{bmatrix}
1,1 & 0 & 1,3 & 0 \\
0 & 0 & 3,1 & 0 \\
-1 & 0 & 3,3 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
d_{1x} \\
d_{1y} \\
d_{3x} \\
d_{3y}
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_{1x} \\
F_{1y} \\
F_{2x} \\
F_{2y} \\
F_{3x} \\
F_{3y}
\end{bmatrix} = \frac{EA}{L}
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
d_{1x} \\
d_{1y} \\
d_{2x} \\
d_{2y} \\
d_{3x} \\
d_{3y}
\end{bmatrix}
\]
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\[
\begin{align*}
\begin{bmatrix}
  f_{2x} \\
  f_{2y} \\
  f_{3x} \\
  f_{3y}
\end{bmatrix} &= \frac{EA}{L} \begin{bmatrix}
  \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\
  -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
  -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
  \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}
\end{bmatrix} \begin{bmatrix}
  d_{2x} \\
  d_{2y} \\
  d_{3x} \\
  d_{3y}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
  F_{1x} \\
  F_{1y} \\
  F_{2x} \\
  F_{2y} \\
  F_{3x} \\
  F_{3y}
\end{bmatrix} &= \frac{EA}{L} \begin{bmatrix}
  1 & 0 & -1 & 0 \\
  0 & 0 & 0 & 0 \\
  \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
  -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\
  -1 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\
  0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}
\end{bmatrix} \begin{bmatrix}
  d_{1x} \\
  d_{1y} \\
  d_{2x} \\
  d_{2y} \\
  d_{3x} \\
  d_{3y}
\end{bmatrix}
\end{align*}
\]
\[
\begin{align*}
\begin{bmatrix}
f_{1x} \\
f_{1y} \\
f_{2x} \\
f_{2y}
\end{bmatrix} &= \frac{EA}{L} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
d_{1x} \\
d_{1y} \\
d_{2x} \\
d_{2y}
\end{bmatrix}
\end{align*}
\]
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The vector of nodal forces in the global coordinate system is given by:

\[
\begin{bmatrix}
F_{1x} \\
F_{1y} \\
F_{2x} \\
F_{2y} \\
F_{3x} \\
F_{3y}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\
0 & -1 & -\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
-1 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}
\end{bmatrix}
\begin{bmatrix}
d_{1x} \\
d_{1y} \\
d_{2x} \\
d_{2y} \\
d_{3x} \\
d_{3y}
\end{bmatrix}
\]

where \( \{d\} \) is the vector of nodal displacements in the global coordinate system, \( \{F\} \) is the vector of nodal forces in the global coordinate system, \( \{dK\} \) is the vector of nodal displacements in the component stiffness coordinate system, and \( \{d\} \) is the vector of nodal displacements in the global coordinate system.

\[
\{F\} = [K] \{d\}
\]

The component stiffness matrix is given by:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\
0 & -1 & -\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
-1 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}
\end{bmatrix}
\]

This matrix relates the forces at each node to the displacements of the nodes in the global coordinate system.
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- How to deal with the problem of supports (restraints)?
- These are nodes where the displacements are known, zero in the perfectly rigid support case.

\[
\begin{bmatrix}
F_{1x} \\
F_{1y} \\
F_{2x} \\
F_{2y}
\end{bmatrix} = \frac{EA}{L} \begin{bmatrix}
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & \sqrt{2}/4 & \sqrt{2}/4 \\
0 & -1 & -\sqrt{2}/4 & 1+\sqrt{2}/4 \\
\end{array}
\end{bmatrix} \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{2}/4 & 1+\sqrt{2}/4 \\
0 & 0 & \sqrt{2}/4 & -\sqrt{2}/4 \\
0 & 0 & \sqrt{2}/4 & \sqrt{2}/4 \\
\end{bmatrix} \begin{bmatrix}
d_{3x} \\
d_{3y}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

-unknown support Reactions
- known support Displacements
- known applied nodal loads
- unknown nodal Displacements
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With the stiffness matrix partitioned as follows

\[
\begin{bmatrix}
  f_{1x} \\
  f_{1y} \\
  f_{2x} \\
  f_{2y} \\
  0 \\
  -P
\end{bmatrix} = \frac{EA}{L} \begin{bmatrix}
  K_{11} & K_{12} \\
  K_{21} & K_{22}
\end{bmatrix} \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  d_{3x} \\
  d_{3y}
\end{bmatrix}
\]

where

\[
K_{11} = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & -1 & 0 \\
  0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
  0 & 1 & -\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4}
\end{bmatrix}
\]

\[
K_{12} = \begin{bmatrix}
  -1 & 0 \\
  0 & 0 \\
  -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\
  \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}
\end{bmatrix}
\]
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\[
K_{21} = \begin{bmatrix}
-1 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\
0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}
\end{bmatrix}
\]

\[
K_{22} = \begin{bmatrix}
1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
-\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}
\end{bmatrix} = S_j
\]

then determine nodal displacements first by solving the following system of equations:

\[
\begin{bmatrix}
0 \\
-P
\end{bmatrix} = [K_{22}]^{-1} \begin{bmatrix}
d_{3x} \\
d_{3y}
\end{bmatrix} = \frac{L}{EA} \begin{bmatrix}
1 + \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
-\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
-P
\end{bmatrix} = \frac{PL}{AE} \begin{bmatrix}
-1 \\
-1 - 2\sqrt{2}
\end{bmatrix}
\]
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Next determine the nodal forces:

\[
\begin{pmatrix}
F_{1x} \\
F_{1y} \\
F_{2x} \\
F_{2y} \\
0 \\
-P
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\
0 & -1 & -\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
-1 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\
0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4}
\end{pmatrix}
\begin{pmatrix}
\frac{EA}{L} \\
\frac{PL}{AE}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
d_{3x} = -1 \\
d_{3y} = -1 - 2\sqrt{2}
\end{pmatrix}

= \begin{pmatrix}
P \\
0 \\
-P \\
P \\
0 \\
-P
\end{pmatrix}
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In general, with

\[ \{ \hat{d} \} = [T] \{ d \} \]

\[
\begin{align*}
\hat{d}_1 &= \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{1y} \end{bmatrix} \\
\hat{d}_2 &= \begin{bmatrix} 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} d_{2x} \\ d_{2y} \end{bmatrix}
\end{align*}
\]

The for element #1 in the truss

\[
\begin{align*}
\begin{bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{3x} \\ \hat{d}_{3y} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{PL}{AE} \\ -1 \end{bmatrix} \\
\begin{bmatrix} \hat{d}_{3x} \\ \hat{d}_{3y} \end{bmatrix} &= \begin{bmatrix} - \frac{PL}{AE} \\ \frac{(-1-2\sqrt{2})PL}{AE} \end{bmatrix}
\end{align*}
\]
The strain in element #1 is

\[ \varepsilon(x) = \{B\} \{d\} \]

\[ = \begin{bmatrix} -\frac{1}{L} & 1 \end{bmatrix} \begin{bmatrix} -\frac{PL}{AE} \\ \frac{(-1-2\sqrt{2})PL}{AE} \end{bmatrix} \]

\[ = -\frac{2\sqrt{2}P}{EA} \]

Negative sign indicates compression. Calculate the stress and the force in element #1

\[ \sigma(x) = E \varepsilon(x) \]

\[ = -\frac{2\sqrt{2}P}{A} \]

\[ F(x) = \sigma(x)A \]

\[ = -2\sqrt{2}P \]
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In class example
Potential Energy Approach in Deriving Bar Element Equations

The principle of minimum potential energy is now used to derive the bar element equations presented earlier. Recall that

\[ \pi_p = U + \Omega \]

For the internal strain energy component consider the following figure:
The differential of strain energy is given by the expression:

\[
dU = \sigma_x (\Delta y) (\Delta z) (\Delta x) d\varepsilon_x = \sigma_x d\varepsilon_x dV
\]

For the entire bar:

\[
U = \int_V \left\{ \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x \right\} dV = \int_V \left\{ \int_0^{\varepsilon_x} (E\varepsilon_x) d\varepsilon_x \right\} dV
\]

\[
= \int_V \left\{ \int_0^{\varepsilon_x} \frac{E(\varepsilon_x)^2}{2} \right\} dV = \int_V \left\{ \frac{(E\varepsilon_x)(\varepsilon_x)}{2} \right\} dV = \frac{1}{2} \int_V \sigma_x \varepsilon_x dV
\]
The potential energy term is associated with externally applied loads and is given by the expression

\[
\Omega = -\int_V \dot{X}_b \dot{u} \, dV - \int_S \dot{T}_x \dot{u} \, dS - \sum_{i=1}^{M} f_{ix} \hat{d}_{ix}
\]

where the first term represents contributions from any body forces, the second term represents any contribution from surface tractions, and the third term represents the contribution from concentrated nodal forces.

The total potential energy for a bar element of length L and constant cross sectional area A becomes:

\[
\pi_p = U + \Omega \\
= \frac{1}{2} \int_V \sigma_x \varepsilon_x \, dV - \int_V \dot{X}_b \dot{u} \, dV - \int_S \dot{T}_x \dot{u} \, dS - \sum_{i=1}^{M} f_{ix} \hat{d}_{ix} \\
= \frac{A}{2} \int_0^L \sigma_x \varepsilon_x \, d\hat{x} - \int_V \dot{X}_b \dot{u} \, dV - \int_S \dot{T}_x \dot{u} \, dS - f_{1x} \hat{d}_{1x} - f_{2x} \hat{d}_{2x}
\]
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With

\[ \sigma(x) = -\frac{P}{A} \]
\[ = E\varepsilon(x) \]
\[ = [D]\varepsilon(x) \]

and

\[ [D] = [E] \]

then formally in matrix notation

\[ \{\sigma_x\} = [D]\{\varepsilon_x\} \]

\[ \pi_p = \frac{A}{2} \int_0^L \{\sigma_x\}^T \{\varepsilon_x\} d\hat{x} - \int_V \{\hat{u}\}^T \{\hat{X}_b\} dV - \int_S \{\hat{u}\}^T \{\hat{T}_x\} dS - P \hat{d}_{1x} - P \hat{d}_{2x} \]
\[ = \frac{A}{2} \int_0^L \{\sigma_x\}^T \{\varepsilon_x\} d\hat{x} - \int_V \{\hat{u}\}^T \{\hat{X}_b\} dV - \int_S \{\hat{u}\}^T \{\hat{T}_x\} dS - \{\hat{d}\}^T \{P\} \]
which leads to

\[
\pi_p = \frac{A}{2} \int_0^L \left( \{ \sigma_x \}^T = \{ \hat{d} \}^T [B] [D]^T \right) \left( \{ \varepsilon_x \} = [B] \{ \hat{d} \} \right) \, d\hat{x} \\
- \int_v \left( \{ \hat{u} \}^T = \{ \hat{d} \}^T [N]^T \right) \{ \hat{X}_b \} \, dV - \int_s \left( \{ \hat{u} \}^T = \{ \hat{d} \}^T [N]^T \right) \{ \hat{T}_x \} \, dS \\
- \{ \hat{d} \} \{ P \}
\]

or

\[
\pi_p = \frac{A}{2} \int_0^L \{ \hat{d} \}^T [B]^T [D]^T [B] \{ \hat{d} \} \, d\hat{x} - \int_v \{ \hat{d} \}^T [N]^T \{ \hat{X}_b \} \, dV \\
- \int_s \{ \hat{d} \}^T [N]^T \{ \hat{T}_x \} \, dS - \{ \hat{d} \} \{ P \}
\]
Integration yields

\[ \pi_p = \frac{A}{2} \left\{ \hat{d} \right\}^T [B]^T [D]^T [B] \left\{ \hat{d} \right\} \int_0^L d\hat{x} \]

\[ - \left\{ \hat{d} \right\}^T \left[ \int_V [N]^T \left\{ \hat{X}_b \right\} dV - \int_S [N]^T \left\{ \hat{T}_x \right\} dS - \{ P \} \right] \]

\[ = \frac{AL}{2} \left\{ \hat{d} \right\}^T [B]^T [D]^T [B] \left\{ \hat{d} \right\} - \left\{ \hat{d} \right\}^T \left\{ \hat{f} \right\} \]

where

\[ \left\{ \hat{f} \right\} = \int_V [N]^T \left\{ \hat{X}_b \right\} dV + \int_S [N]^T \left\{ \hat{T}_x \right\} dS + \{ P \} \]

now one can definitely see that

\[ \pi_p = \pi_p \left( d_{1x}, d_{2x} \right) \]

\[ = \pi_p \left( \left\{ \hat{d} \right\} \right) \]
The minimization of potential energy requires

\[
\frac{\partial \pi_p}{\partial \hat{d}_{1x}} = 0 \quad \frac{\partial \pi_p}{\partial \hat{d}_{2x}} = 0
\]

With

\[
\{U^*\} = \{\hat{d}\}^T [B]^T [D]^T [B]\{\hat{d}\}
\]

\[
= \begin{pmatrix} \hat{d}_{1x} & \hat{d}_{2x} \end{pmatrix} \begin{pmatrix} -1 \\ \frac{-1}{L} \\ \frac{1}{L} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ \frac{-1}{L} & \frac{1}{L} \end{pmatrix} \begin{pmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{pmatrix}
\]

\[
= \frac{E}{L^2} \left( \hat{d}_{1x}^2 - 2\hat{d}_{1x}\hat{d}_{2x} + \hat{d}_{2x}^2 \right)
\]
and

\[ \{ \hat{d} \}^T \{ \hat{f} \} = \hat{d}_{1x} \hat{f}_{1x} + \hat{d}_{2x} \hat{f}_{2x} \]

then

\[ 0 = \frac{\partial \pi_p}{\partial \hat{d}_{1x}} = \frac{\partial}{\partial \hat{d}_{1x}} \left[ \frac{AL}{2} \frac{E}{L^2} \left( \hat{d}^2_{1x} - 2\hat{d}_{1x} \hat{d}_{2x} + \hat{d}^2_{2x} \right) - \left( \hat{d}_{1x} \hat{f}_{1x} + \hat{d}_{2x} \hat{f}_{2x} \right) \right] \]

\[ = \frac{AL}{2} \left[ \frac{E}{L^2} \left( 2\hat{d}_{1x} - 2\hat{d}_{2x} \right) \right] - \hat{f}_{1x} \]

\[ 0 = \frac{\partial \pi_p}{\partial \hat{d}_{2x}} = \frac{\partial}{\partial \hat{d}_{2x}} \left[ \frac{AL}{2} \frac{E}{L^2} \left( \hat{d}^2_{1x} - 2\hat{d}_{1x} \hat{d}_{2x} + \hat{d}^2_{2x} \right) - \left( \hat{d}_{1x} \hat{f}_{1x} + \hat{d}_{2x} \hat{f}_{2x} \right) \right] \]

\[ = \frac{AL}{2} \left[ \frac{E}{L^2} \left( -2\hat{d}_{1x} + 2\hat{d}_{2x} \right) \right] - \hat{f}_{2x} \]
These last two equations can be put into a matrix format as follows

\[
\frac{\partial \pi_p}{\partial \{ \hat{d} \}} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{bmatrix} - \begin{bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

or

\[
\begin{bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{bmatrix}
\]

where

\[
[k] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

as before.
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In class example
Comparison of a Finite Element Solution to an Exact Solution

It is time to make comparisons. In the previous in-class problem we solved the problem using first one and then two elements. Here we will make comparisons of two, four and eight element models to an exact solution. The exact solution for axial displacements is obtained from the solution of the following expression:

\[ u(x) = \frac{1}{AE} \int P(x) \, dx \]

with

\[ P(x) = \left(\frac{1}{2}\right)x(10x) = 5x^2 \]
then

\[ u(x) = \frac{1}{AE} \int 5x^2 \, dx \]

\[ = \frac{5x^3}{3AE} + C_1 \]

Evaluating this expression at the support yields

\[ u(x = L) = 0 \]

\[ = \frac{5L^3}{3AE} + C_1 \]

or

\[ C_1 = -\frac{5L^3}{3AE} \quad \rightarrow \quad u = \frac{5x^3}{3AE} \left( x^3 - L^3 \right) \]
The following figure plots the exact expression for the displacement along the rod along with the finite element calculations:

The finite element analysis match the exact solutions at the node points. Although the nodal displacements match the exact solution, the displacement values at intermediate locations are poor because we are using linear displacement functions within each element. As we increase the number of elements we converge on the exact solution.
The exact solution for stress is:

\[ \sigma = \frac{P(x)}{A} = E \varepsilon = E \left( \frac{du}{dx} \right) = E \left( \frac{d\delta}{dx} \right) \]

\[ = \frac{5x^2}{2} = 2.5 x^2 \]

Since stress is derived from the slope of the displacement, and since the displacement is linear within each element, then the stress is constant in each element. This can be seen in the following figure:

![Graph showing stress distribution](image)

The best approximation of the stress occurs at the midpoint of the element, not at the nodes. The reason for this is that the derivative of displacement is better predicted between the nodes then at the nodes.
As we saw in the example problem the axial stress is not continuous across element boundaries. Recall that equilibrium is not satisfied within each element, only at the nodes. As the number of elements increase the discontinuity in stress decreases and the approximation of equilibrium improves.

Finally in the figure below the convergence of the axial stress at the fixed end is depicted. Again, as the number of elements increase the stress at the fixed end converges to the exact solution from below.