Short Communication

Numerical integration in the axisymmetric finite element formulation

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Abstract

This paper reported the integration of the element stiffness matrix for an axisymmetric finite element formulation. A survey of the various methodologies found in the literature was discussed. Two numerical integration strategies were presented to overcome the inaccuracy that arose when an element was on or close to the axis of rotation.

Keywords: Finite element; Axisymmetric; Numerical integration; Singular integral; Nearly singular integral

1. Introduction

Axisymmetric finite element (FE) models may be used to represent three-dimensional structures exhibiting symmetry about a central axis of rotation (axis of symmetry). For the conventional axisymmetric elements to be acceptable for modeling a structure, the body’s geometry, loading, boundary conditions and material properties must all be independent of the $u$ coordinate. Common types of axisymmetric elasticity elements include the two- and three-node washers, the three- and six-node triangles, and the four- and eight-node quadrilaterals. Structures commonly modeled using axisymmetric elasticity elements include thick-walled pressure vessels; soil masses subjected to circular footing loads, and flywheels rotating at constant angular velocities.

The axisymmetric element stiffness matrix and the different types of integrands that arise are discussed in Section 2. Section 3 surveys the methods that are used to integrate the axisymmetric element stiffness matrix. Finally, Section 4 discusses two methods for overcoming the numerical inaccuracies that occur when the element lies on and close to the axis of rotation.

2. Element stiffness matrix and integrand types

The stiffness matrix for general axisymmetric elasticity elements is of the following form \cite{1}:

$$K_E = \int \int _V B^T D B r \, dr \, d\theta \, dz,$$

where $B$ is the kinematic matrix that relates the element strains to the element nodal displacements ($e = B u$), $D$ is the material law (Hooke’s law in this case) that relates the element stresses to the element strains ($\sigma = D e$), superscript $T$ denotes the transpose operation, and $r \, dr \, d\theta \, dz$ is the element volume $V$. In the axisymmetric elements, the hoop strain $\varepsilon_r$ is a function of $1/r$ that varies with the radial position in the element. For this reason, $B$ contains $1/r$ terms, as does the integrand $B^T D B r$.

The integrals that arise from Eq. (1) are classified as regular, singular \cite{2}, or nearly singular \cite{3,4} as shown in Fig. 1 for a two-node washer element. A regular (or proper) integral has finite lower and upper limits of integration, and its integrand remains finite over these limits. A singular (or improper) integral may have at least one of its integration limits as infinite ($^\dagger$) or its integrand may become infinite at one or more points within the finite limits of integration. A nearly singular (or quasi-singular) integral has an integrand that behaves very strongly or changes rapidly near one of the integration limits, but does not diverge. In a strict mathematical sense, nearly singular integrals are no different from regular integrals; however, nearly singular integrals are not always handled correctly by numerical integration (as discussed later).

As the integrand $B^T D B r$ is integrated over the radial coordinate, the integral in (1) becomes improper as $r$ approaches zero. For elements on the axis of rotation, the integral is either singular or improper; and for those that are
very close to the axis of rotation, the lower limit of integration for the radial coordinate is almost zero, and the integral is nearly singular (as 1/r behaves very strongly near r = 0). For elements farther from the axis, the integral is regular or proper.

3. Survey of methods for integrating the element stiffness matrix

Explicit integration may be used to evaluate (1) for two-node washer, three-node triangle and four-node rectangular elements. For the general four-node quadrilateral elements, the integrand in (1) includes terms that are ratios of polynomial functions; hence, explicit integration cannot be used for these elements. When explicit integration is used, the element stiffness matrix may be evaluated exactly. For the regular and nearly singular cases, conventional integration methods such as the chain rule from [2] may be applied. For the singular case, where an element has node(s) on the axis of rotation, the stiffness matrix is evaluated in the limit as the radial coordinate of these node(s) approaches zero, with indeterminate forms handled using L’Hôpital’s rule. Symbolic software codes such as Maple [5] may be used to integrate (1) explicitly for the simple washer, triangular and rectangular elements; however, explicit integration is not commonly used in practice as it typically involves much more computational effort than numerical integration. Further, explicit integration cannot be carried out for the six-node triangle and the eight-node quadrilateral.

In the axisymmetric models a mean radius approximation is often used, in which the $B^T DBr$ product is evaluated at the element’s centroid prior to integration [1,6–9]. In other words, $r$ is replaced by a constant, $r_m$, in the integrand in (1). This method is suggested for both triangular [1,7–9] and washer elements [6], but never for quadrilateral elements as its stiffness matrix is rank deficient [6]. Fenner [8] and Gallagher [9] while recommending the mean radius method caution that the accuracy for this approximation deteriorates for elements near the axis of rotation. This method is inaccurate for the singular and nearly singular cases because the 1/r terms in $B$ vary rapidly near the axis of rotation. This approximation is equivalent to a one-point rule using standard Gauss quadrature.

Numerical integration is the most widely used means to evaluate the element stiffness matrix. Gauss (or Gauss–Legendre) quadrature [1,6,10] is more accurate and requires less computational effort than other routines such as the Newton–Cotes method [10]; therefore it is the usual method used for the integration in (1). The general element stiffness matrix can be written in the numerical integration form from Eq. (1) as follows

$$K_E = \int_0^{2\pi} \int_{-1}^{+1} \int_{-1}^{+1} B^T(s,t) DB(s,t) r(s,t) |J(s,t)| ds \, d\theta$$

$$= 2\pi \sum_{i=1}^{n} \sum_{j=1}^{n} B^T(s_i,t_j) DB(s_i,t_j) r(s_i,t_j) |J(s_i,t_j)| w_i w_j,$$

where $J$ is the Jacobian matrix, $\mid \mid$ denotes the determinate, $n$ the number of Gauss points, $i$ and $j$ the Gauss point and $w$ the weight. Eq. (2) states that the integrand is evaluated and multiplied by a weight at pre-selected sampling points. The weighted values are summed to yield, in general, an approximation of the integral. A Gauss quadrature rule using $n$ sampling points integrates exactly a polynomial of up to order $2n - 1$. Gauss quadrature is inexact for the integrand in (2), which is not a polynomial because of the 1/r terms. In the finite element literature, only the regular and singular cases are addressed. However, in most instances the singular case is ignored and the nearly singular case addressed only in one instance.

In isoparametric formulations, the standard practice is to use a two-point quadrature rule for two-node line elements and a 2 x 2 quadrature rule for four-node quadrilateral elements [6,10]. However, the authors believe that these conventions were developed for one-dimensional, linear bar and two-dimensional plane stress or plane strain cases, respectively, and not specifically for axisymmetric elements, which are unique because of the 1/r terms. Expression (2) for axisymmetric elements far from the axis may be integrated very accurately using the Gauss quadrature as the 1/r function changes slowly as $r$ varies for large values of $r$ and may be accurately approximated by a polynomial as shown in Fig. 1(a). The rules for higher order elements are stated in Refs. [6,10] and the comments discussed above still apply.
Gauss quadrature is inappropriate for evaluating diverging singular integrals because this numerical technique always yields a finite value for the integral, even if the integral diverges when evaluated explicitly. Gauss quadrature may also be ineffective for evaluating converging singular integrals [3], with its effectiveness dependent upon the specific form of the integral. Despite these problems, few authors of FE textbooks address the singularity at the axis of rotation in axisymmetric formulations. Most of them recommend using conventional Gauss quadrature to integrate the stiffness matrix, regardless of element location. Many authors [6,11–14] argue that standard quadrature may be employed effectively as the sampling points are not located along the axis of rotation, and thus the integrand is not evaluated at \( r = 0 \) (hence the \( 1/r \) terms remain finite). However, applying the conventional two-point (1D) or 2 \( \times \) 2 (2D) Gauss quadrature rules for singular elements may lead to significant errors in the solution.

A few authors do discuss the singular case in more detail. Burnett [12] mentions using special “core elements” with modified displacement functions to eliminate the singular terms in the stiffness matrix integrand for elements on the axis of rotation. Presumably these core elements may be used in a mesh of triangular or quadrilateral elements; Burnett [12] does not specify that the core elements must be used in a mesh of elements of a particular type. Gallagher [9] and Yang [15] propose, in order to avoid dealing with diverging \( 1/r \) terms, using special rectangular core elements near the axis of rotation in a mesh otherwise made up of triangular elements.

For the singular case, many authors recommend prescribing the boundary condition of zero radial displacement for all nodes on the axis of rotation [6,9,12–14]. This prevents formation of a pinhole or overlap in the mesh (material continuity). Applying this boundary restraint also results in the elimination of all the diverging integral terms in the global stiffness matrix \( K \), before the global \( f = Ku \) equation is solved for the nodal displacements. However, the authors of this paper have found that applying this boundary condition does not eliminate the possible errors due to other numerically integrated entries in the stiffness matrix, which do not diverge but contain \( 1/r \) terms (converging singular integrals). In addition, this boundary condition cannot be applied for the nearly singular case, as there are no nodes on the axis of rotation and a hole already exists in the material at \( r = 0 \).

Standard Gauss quadrature also often yields incorrect results for nearly singular integrals, especially when only a few sampling points are used [3]. Therefore, the usual two-point (1D) or 2 \( \times \) 2 (2D) Gauss quadrature conventions may be insufficient for evaluating the integral in (2) for nearly singular elements. However, no FE textbooks present special rules for addressing the nearly singular case for axisymmetric formulations; the conventional Gauss quadrature rules are recommended for all elements, regardless of the proximity to the axis of rotation. Only Cook [6] suggests using more sampling points in the standard Gauss quadrature routine for elements close to the axis of rotation. However, this reference does not specify how many additional points should be used or how close the elements must be to the axis to merit the use of these extra points.

### 4. Methods to overcome singular and nearly singular integrals

As the conventional Gauss quadrature often yields inaccurate results for singular and nearly singular integrations, various modified quadrature rules have been developed for evaluating these kinds of integrals. One such modified Gauss quadrature rule developed by Telles [3,4] employed a cubic transformation. The cubic transformation improves the accuracy of standard Gauss quadrature for convergent singular and nearly singular integrals by shifting the sampling points closer to the location of singularity, thus accounting for the strong behavior of the integrand in the proximity of the singular point (Fig. 2). When the transformation is used for an element, the sampling points move closer to \( r = 0 \) as the integral becomes more nearly singular. For regular integrals, the transformation degenerates to conventional Gauss quadrature, with the usual sampling points; therefore, the transformation technique may be safely applied to all elements in a mesh, whether regular, singular, or nearly singular. When applied to a convergent
singular integral, the Gauss point locations are transformed closer towards the singularity. One must remember to set the radial displacement to zero at nodes on the axis of rotation, to eliminate the diverging singular integrals. The transformation technique can be applied to the element stiffness matrix in Eq. (2). The entries in the $B^TDBr$ product are then divided into, e.g. constant terms, terms linear in $r$, and terms of the order $1/r^2$ (type and number of terms depend on the element under consideration). The constant and linear terms are integrated using conventional Gauss quadrature, each evaluated at the usual Gauss–Legendre sampling points. The $1/r$ terms are evaluated at the transformed (Telles) sampling points and are then multiplied by the Jacobian of the cubic transformation (different from the Jacobian based on conventional quadrature). The location of the singularity in the $s$-domain (required to employ the transformation) is found by setting $r = 0$ in (2). The authors caution that the transformed sampling points found using the singularity at $r = 0$ may not be applied to every term in every entry of the $B^TDBr$ matrix. Only the terms of order $1/r$ are not defined at the singularity point and can be evaluated at the transformed points. Thus, a major drawback in applying this technique is that the transformation must be used selectively on certain terms in the expanded $B^TDBr$ product. Symbolically, expanding the $B^TDBr$ integrand for such complicated elements such as the four-node quadrilateral and applying the rule to all the $1/r$ terms is very tedious. The authors suggest that all terms in the integrand, whether requiring the conventional Gauss or the transformed sampling points should be evaluated using the same number of sampling points (the same order of quadrature). Applying the rules found in Refs. [6,10] for the various element types in conjunction with the Telles transformation will improve the accuracy of the element stiffness matrix.

An FE analyst/programmer may not prefer to employ the Telles transformation technique as the computer program must be rewritten. In other words, the current FEM programs evaluate the $B(s_i, t_j)$ and $|J(s_i, t_j)|$ at each Gauss point and then the stiffness matrix at the corresponding point is determined by evaluating $2\pi r B^T(s_i, t_j) \cdot DB(s_i, t_j) r(s_i, t_j) |J(s_i, t_j)| w_i w_j$. The element stiffness matrix is then determined by summing up all the Gauss point element stiffness matrices. A strategy is proposed so that minimal effort is needed to modify the program. As the function $1/r$ varies significantly as an element nears or lies on the axis of rotation (Fig. 1(b) and (c)), a higher order Gauss quadrature rule is required for nearly singular and singular integrands. Determining the strength of integrand can be based upon a non-dimensional radial distance. The distance is normalized in terms of the ratio $r_{md}/l_T$, where $r_{md}$ is the radial coordinate (in the global coordinate system) of the centroid of the element (the midpoint of the innermost element for the washer) and $l_T$ is the length of the element (the integration domain) in the radial direction. A $r_{md}/l_T$ value of 0.5 denotes a singular element, with a node at $r = 0$. Nearly singular elements have $r_{md}/l_T$ values slightly greater than 0.5. Therefore, for a problem with a small pinhole, the FE code would simply increase the order of Gaussian integration for elements on or close to the axis of rotation.

5. Conclusion

Current numerical integration strategies are inadequate in evaluating singular and nearly singular integrals that arise in axisymmetric finite element formulation. Two numerical integration strategies were proposed to increase the accuracy of the element stiffness matrix when an element is close to the axis of rotation or lies on the axis of rotation. The first is based on the cubic transformation proposed by Telles [3,4], while the second is based on the increasing order of Gauss quadrature for an element that lies on or near the axis of rotation. An in-depth discussion of the work presented in this paper can be found in Ref. [16].

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References