STRESSES IN ROTATING DISKS OF MATERIALS WITH DIFFERENT COMPRESSIVE AND TENSILE MODULI

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Summary—The problem of rotating disks of bi-linear elastic materials is treated herein. Stress and deformation states in the disk are obtained and are compared with linear elastic solutions.

NOTATION

\[ r \] radial co-ordinate of the disk
\[ \sigma_r, \sigma_\theta \] radial and tangential stresses
\[ \rho \] mass density
\[ \Omega \] angular velocity of rotation
\[ \Phi \] stress function
\[ u_r \] radial displacement
\[ \varepsilon_r, \varepsilon_\theta \] radial and tangential strains
\[ E_t, E_c \] Young’s moduli in tension and compression
\[ \nu_t, \nu_c \] Poisson’s ratios in tension and compression
\[ a \] radius of the disk
\[ r_1, r_2 \] radial co-ordinates of boundaries of different stress states

INTRODUCTION

Among the various engineering materials which show non-linear behaviour and are modelled by non-linear stress-strain relations, a particular class has gained special attention. It is known that a wide variety of materials show a marked difference in behaviour when subjected to compressive and tensile forces. The material bodies in which this property is manifested can be categorized into the classes of natural and man-made composites. Among the natural materials having this character one can cite all types of wood, rocks, soils and their products. Moreover, biological bodies whose material and geometrical structure is determined by their function depict the property of having different strengths in compression and in tension [see e.g., ref. (1)]. In addition to natural composites, many of the man-made fiber-reinforced or laminated composites have different compressive and tensile strengths; there exists some experimental evidence that materials such as polycrystalline graphites,\textsuperscript{3} cord–rubber laminates\textsuperscript{3} and structural masonry\textsuperscript{4} can be considered to have a “bi-modulus” material character. An idealized model of elastic materials with different compressive and tensile moduli, according to which the stress-strain relation can be subdivided into linear parts has been considered by Ambartsomian and other investigators [see e.g., refs. (5)–(7)]. Based on such a bi-linear model several static and stability problems of engineering interest have also been solved [see e.g., refs. (5)–(9)].
It is appropriate to point out, in this connection, that a degenerate model of bi-modulus materials which, to a fair degree of accuracy, represents the behaviour of a certain class of bodies is a material with no tensile strength. Problems of stability, deformation, load-carrying capacity and restoring force property of such materials have been treated by various investigators (see for example refs. (10)–(13)). Reference (14) is an extension of the latter type of problems for bi-modulus restoring forces.

In the present paper we shall consider the problem of a rotating disk made of materials with different compressive and tensile moduli. Although the consideration of such a problem may appear to be a somewhat restricted choice, it is meant to exemplify a wider class of problems of rotating bodies of various geometries and types. The specific results obtained in this paper, however, can be applied to predict the stress and deformation states in rotating disks of bi-modulus materials and solid particles subjected to centrifugal action. It is to be added that the consideration of the steady-state rotational motion of a bilinear disk is also aimed at representing an attempt toward the study of dynamical problems relating to these materials.

**Rotating Disks of Bi-Modulus Materials**

Consider a thin circular disk which is rotating about an axis, perpendicular to it at its centre, with a constant angular velocity.

The stress equation of motion of such a disk is written as

\[ \frac{d}{dr} (r \sigma_r) - \sigma_\theta + \rho \Omega^2 r^2 = 0. \]  

(1)

This equation is satisfied by a stress function \( F \) defined by the following relations:

\[ r \sigma_r = F \quad \text{and} \quad \sigma_\theta = \frac{dF}{dr} + \rho \Omega^2 r^2. \]  

(2)

The kinematical relations for the problem have the form

\[ \varepsilon_r = \frac{du_r}{dr} \quad \text{and} \quad \varepsilon_\theta = \frac{u_\theta}{r}, \]  

(3)

which yield the following compatibility equation

\[ \varepsilon_\theta - \varepsilon_r + r \frac{d\varepsilon_\theta}{dr} = 0. \]  

(4)

The constitutive relations for a disk of material with different moduli in compression and tension can be written as

\[ \varepsilon_r = a_{11} \sigma_r + a_{12} \sigma_\theta \]  

(5)

and

\[ \varepsilon_\theta = a_{21} \sigma_r + a_{22} \sigma_\theta. \]  

(6)

where \( a_{11}, a_{12}, a_{21} \) and \( a_{22} \) are elastic compliances of the bi-modulus material and are dependent on the state of the stress in the body. Accordingly, we have

I. \( \sigma_r > 0, \quad \sigma_\theta > 0, \)

\[ a_{11} = a_{22} = \frac{1}{E_t}, \quad a_{12} = a_{21} = -\frac{\nu_t}{E_t}. \]  

(7)

II. \( \sigma_r < 0, \quad \sigma_\theta > 0, \)

\[ a_{11} = \frac{1}{E_c}, \quad a_{12} = a_{21} = -\frac{\nu_t}{E_t} = -\frac{\nu_c}{E_c}, \quad a_{22} = \frac{1}{E_t}. \]  

(8)
III. \( \hat{\sigma}_r < 0, \quad \hat{\sigma}_\theta < 0, \)

\[
a_{11} = a_{22} = \frac{1}{E_c}, \quad a_{12} = a_{21} = -\frac{\nu_t}{E_t} = -\frac{\nu_c}{E_c}.
\]  

(9)

Note that in the above relations the tensile and compressive stresses are taken to have positive and negative signs, respectively.

Relations (1)–(9) represent a complete set of field equations for spinning disks with different compressive and tensile moduli. To complete the formulation of the problem these equations must be accompanied by an appropriate set of boundary conditions.

It is observed that the problem is essentially a non-linear one, with its non-linearity arising from a bi-linear material behaviour. However, the fact that the physical domain can be subdivided into various regions, in each of which a set of linear governing equation is dominant, renders the problem to manageable analytical treatment.

One complication remains, however, namely that the boundaries of various regions are not known \textit{a priori} and are dependent on the state of stress and deformation in the body; they are to be determined along with the rest of the unknowns of the problem.

**SOLUTION OF THE PROBLEM**

As a case of technical importance we consider a solid disk of bimodulus material which has zero radial displacement at the outer boundary. It was pointed out that for analytical treatment of the problem, the domain must be divided into regions. To gain an \textit{a priori} knowledge of possible regions of stress states that can exist the linear problem of an elastic disk with equal moduli in compression and tension is considered. For such a disk, \( E_c = E_t = E \) and \( \nu_c = \nu_t = \nu \). Solution for the latter problem can set a useful guideline for subdividing the disk into various regions. For an elastic disk with equal moduli in compression and tension we have the following regions of stress states:

(a) Region I, \( \sigma_r > 0, \quad \sigma_\theta > 0, \quad 0 \leq r \leq r_1. \)

(b) Region II, \( \sigma_r < 0, \quad \sigma_\theta > 0, \quad r_1 \leq r \leq r_2. \)

(c) Region III, \( \sigma_r < 0, \quad \sigma_\theta < 0, \quad r_2 \leq r \leq a. \)

Now, to obtain the state of stress and deformation in a rotating disk of bi-modulus materials we write the solution to the equations (1)–(9) in each of the above regions. Utilizing the boundary conditions and by matching the solution at the interfaces we shall obtain the solution in each region and the unknown interfaces at which a region changes. For convenience let us introduce the following dimensionless variables:

\[
\xi = \frac{r}{a}, \quad \xi_1 = \frac{r_1}{a}, \quad \xi_2 = \frac{r_2}{a},
\]

\[
u = \frac{u_r}{a}, \quad \sigma_r = \frac{\hat{\sigma}_r}{E_t}, \quad \sigma_\theta = \frac{\hat{\sigma}_\theta}{E_t},
\]

\[F = \frac{F_0}{aE_t} \quad \text{and} \quad \omega^2 = \frac{\rho \Omega^2 a^4}{E_t}.
\]

(10)

The solutions to equations (1)–(9), in various regions, are as follows:

**Region I, \( \sigma_r > 0, \quad \sigma_\theta > 0 \):**

\[
F = a_1 \xi + a_2 \frac{3 + \nu_t}{8} \omega^2 \xi^3,
\]

(11)

\[
\sigma_r = a_1 + a_2 \frac{3 + \nu_t}{8} \omega^2 \xi^2, \quad 0 \leq \xi \leq \xi_1,
\]

(12)

\[
\sigma_\theta = a_1 - a_2 \frac{1 + 3 \nu_t}{8} \omega^2 \xi^2
\]

(13)

and

\[
u = \xi(1 - \nu_t) \frac{a_1 - (1 + \nu_t) \frac{a_2}{\xi^2} - \frac{1 - \nu_t^2}{8} \omega^2 \xi^2}{\xi^2}. \]

(14)
To obtain the boundary of the first region we set \( \sigma_0 = 0 \) at \( \xi_1 \), so that

\[
\alpha_2 - \frac{\alpha_2}{\xi_1^2} \frac{3 + \nu_t}{8} \omega^2 \xi_1^2 = 0. \tag{15}
\]

**Region II, \( \sigma_0 \leq 0, \sigma_\theta \geq 0 \):**

\[
F = \alpha_3 \xi^3 + \alpha_4 \xi^{r-1} - \frac{3 + \nu_t}{g - (E_t/E_c)} \omega^2 \xi^2, \quad s^2 = \frac{E_t}{E_c},
\]

\[
\sigma_r = \alpha_3 \xi^{r-1} + \alpha_4 \xi - \frac{3 + \nu_t}{g - (E_t/E_c)} \omega^2 \xi^2, \quad \xi_1 \leq \xi \leq \xi_2,
\]

\[
\sigma_\theta = \alpha_3 s^2 \xi^{r-1} - \alpha_4 s^2 \xi - \frac{(E_t/E_c)(1 + 3\nu_t)}{g - (E_t/E_c)} \omega^2 \xi^2
\]

and

\[
u = \xi \left( (s - \nu_t) \alpha_3 \xi^{r-1} - (s + \nu_t) \alpha_4 \xi - \frac{(E_t/E_c)(1 + 3\nu_t)}{g - (E_t/E_c)} \omega^2 \xi^2 \right). \tag{19}
\]

To obtain the outer boundary of the second region we set \( \sigma_\theta = 0 \) at \( \xi_2 \), so that

\[
\alpha_2 s^2 \xi^{r-1} - \alpha_3 s^2 \xi - \frac{(E_t/E_c)(1 + 3\nu_t)}{g - (E_t/E_c)} \omega^2 \xi^2 = 0. \tag{20}
\]

**Region III, \( \sigma_\theta < 0, \sigma_\theta \leq 0 \):**

\[
F = \alpha_5 \xi + \frac{\alpha_5}{\xi} \frac{3 + \nu_t}{8} \omega^2 \xi^2,
\]

\[
\sigma_r = \alpha_5 + \frac{\alpha_5}{\xi^2} \frac{3 + \nu_t}{8} \omega^2 \xi^2, \quad \xi_2 \leq \xi \leq 1,
\]

\[
\sigma_\theta = \alpha_5 \xi^2 - \frac{1 + 3\nu_t}{8} \omega^2 \xi^2
\]

and

\[
u = \xi \left( (1 - \nu_t) \alpha_5 - (1 + \nu_t) \frac{\alpha_5}{\xi^2} - \frac{1 - \nu_t^2}{8} \omega^2 \xi^2 \right). \tag{24}
\]

The boundary and interface conditions are prescribed as follows:

- at \( \xi = 0 \): \( \sigma_r \) is finite, \( \sigma_\theta = 0 \).
- at \( \xi = \xi_1 \): \([\sigma_r, u]_{\text{region I}} = \sigma_\theta \)_{\text{region II}}.
- at \( \xi = \xi_2 \): \([\sigma_r, u]_{\text{region II}} = \sigma_\theta \)_{\text{region III}}.

and

- at \( \xi = 1 \): \( u = 0 \).

Utilization of conditions (15), (20) and (25)–(28) yields eight equations in eight unknowns \( \alpha_1, \ldots, \alpha_5, \xi_1, \xi_2 \).

**NUMERICAL RESULTS AND DISCUSSION**

The equations on the unknowns \( \alpha_1, \ldots, \alpha_5, \xi_1, \xi_2 \) are a set of simultaneous non-linear algebraic equations which are solved numerically by the Newton–Raphson procedure. The dimensionless parameters of the problems to which arbitrary values can be assigned are the dimensionless angular speed of spin, the ratio of tensile and compressive Young’s moduli and compressive (or tensile) Poisson’s ratio. The numerical results obtained are for a range of these parameters which correspond to existing realistic cases of certain bi-modulus materials. Some of the range of parameter variations for which the numerical results to the problem is obtained are listed in Table 1. In all these cases the angular speed was arbitrarily assigned the value, \( \omega = 2 \). The spatial variations of dimensionless radial
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TABLE 1. PARAMETER VARIATION IN NUMERICAL EXAMPLES

<table>
<thead>
<tr>
<th>Curve</th>
<th>$E_c/E_t$</th>
<th>$\nu_c$</th>
<th>$\nu_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.0</td>
<td>0.224</td>
<td>0.224</td>
</tr>
<tr>
<td>B</td>
<td>0.5</td>
<td>0.112</td>
<td>0.224</td>
</tr>
<tr>
<td>C</td>
<td>2.0</td>
<td>0.85</td>
<td>0.425</td>
</tr>
<tr>
<td>D</td>
<td>0.4</td>
<td>0.089</td>
<td>0.222</td>
</tr>
<tr>
<td>E</td>
<td>0.95</td>
<td>0.38</td>
<td>0.40</td>
</tr>
</tbody>
</table>

and tangential stresses and also the radial displacement in the disk, corresponding to those values of parameters, are plotted in Figs. 1–3.

Figs. 1–3 depict a marked difference between the variation of $\sigma_r, \sigma_\theta, u$ in disks of bi-modulus materials (curves B–E) from those quantities corresponding to a linearly

![Fig. 1. Variation of radial stress in a rotating disk of bi-linear materials.](image1)

![Fig. 2. Variation of tangential stress in a rotating disk of bi-linear material.](image2)
elastic material (curve A). The influence of bi-linearity is manifested in changing the shape and magnitudes of displacement and stresses, altering the nodal points at which these quantities change sign and finally introducing discontinuities in the tangential stresses at the interfaces of various regions. Specially, we note that the discontinuity in the tangential stress (although allowed by the uniqueness theorems of elasticity) does not occur in curve A belonging to the linear elastic material of equal moduli. We can verify this latter result by forming the difference of expressions for tangential stresses, in each region, at the interfaces.

From the foregoing results and observations we conclude that the introduction of a more refined and realistic material model can result in a notable change in the elastic behaviour of the body. The problem of rotating disks made of materials with different moduli in compression and tension treated in this paper in addition to being of potential use in interpreting the test results and predicting the behaviour of fiber-reinforced composites, natural composites and other materials which show a dominant bi-linear behaviour, exemplifies a class of problem in which a similar procedure of treatment can be used. Finally, the consideration of a rotating disk problem was partly aimed at providing a step towards the treatment of more complicated dynamical problems concerned with bodies of different anisotropic compressive and tensile moduli.16

REFERENCES