TRANSVERSE SHEAR EFFECT IN A BIMODULUS PLATE

N. KAMIYA
Department of Mechanical Engineering, Mie University, Kamihamacho, Tsu 514, Japan

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This paper concerns the bending analysis of a bimodulus elastic plate whose stress-strain relation is expressed by two straight lines with a slope discontinuity at the origin. An energy formulation of the problem is made, taking into account a transverse shear deformation. A method of numerical solution is presented for cylindrical bending of a plate.

1. Introduction

It is known that some composite materials or high polymers behave differently in simple tension and compression. Certain graphites also exhibit such behavior, which has practical importance in nuclear reactor technology and other fields [1, 2]. Here we restrict our attention to the difference in the linear elastic state of the material. Hence, the difference observed in the nonlinear stress-strain relation of most materials has to be linearized appropriately, i.e. the slightly nonlinear stress-strain relation is approximated by two respective straight lines in tension and compression with a slope discontinuity at the origin (bimodulus material, fig. 1).

Based on the bimodulus nature in simple tension and compression, constitutive equations in the general three-dimensional stress state have been proposed for them by Ambartsumyan and Khachatryan [3] and Ambartsumyan [4]. In these equations elastic constants are related to the sign of the corresponding principal stresses. There are only a few applications of their constitutive equations to stress analyses of structural elements because of the inherent complexity in analysis of the bimodulus material; i.e. the elastic constants involved in the governing equations, which depend on the stress state, are not correctly indicated beforehand. In other words, except in particularly simple problems it is not easy to estimate the stress state of whole elements in the deformed body a priori.

Plate or shell problems in structural analyses are fundamental and important, and a few basic studies have been reported [5-8]. Besides studies on plate or shell theory using the Kirchhoff-Love assumption, Ambartsumyan [6] introduced an effect of transverse shear deformation into axisymmetric deformations of shells of revolution. The problem he dealt with only involved a 'weak moment shell', which meant that, compared to membrane stresses, bending stresses were so small that the stress components did not change their signs along the thickness direction of the shell. Such a peculiar shell is not generally realized, but variation of the sign of stress components in the thickness direction makes analysis difficult.

Although the effect of the transverse shear deformation in the thickness direction may be neglected if a
plate is sufficiently thin, it is remarkable in a thick plate and cannot be disregarded. In particular, it is known that the aforesaid effect in anisotropic plates is more striking than in isotropic plates [9]. The bimodulus isotropic material may be considered as a special orthotropic material whose axes of anisotropy vary with stress states at each material element.

In this paper, as a modification and extension of Ambartsumyan's above-mentioned line of thought [6], a method for the analysis of plate bending is proposed, including the transverse shear deformation in the bimodulus plate by supplementary use of numerical calculation. It is an application of a method similar to that devised by the present author for large deflection of a plate without considering transverse shear [10]. Derivation of the governing equations and a method of calculation are illustrated as an example of cylindrical bending of a moderately thick plate, infinitely long in one direction. Results of numerical calculation for a simple problem show the effect of the transverse shear deformation in the bimodulus plate.

2. Governing equations

Assuming the symmetry of elastic compliance, \( a_{ij} = a_{ji} \) \((i, j = 1, 2, 3)\) the stress–strain relations of bimodulus isotropic materials are expressed in terms of the components of the principal stress directions, \( \alpha, \beta, \gamma \), as follows:

\[
\begin{align*}
\varepsilon_{\alpha} &= a_{11}\sigma_{\alpha} + a_{12}\sigma_{\beta} + a_{12}\sigma_{\gamma}, \\
\varepsilon_{\beta} &= a_{12}\sigma_{\alpha} + a_{22}\sigma_{\beta} + a_{12}\sigma_{\gamma}, \\
\varepsilon_{\gamma} &= a_{12}\sigma_{\alpha} + a_{12}\sigma_{\beta} + a_{33}\sigma_{\gamma},
\end{align*}
\]

(1)

where diagonal elastic compliances \( a_{ii} \) are determined in relation to the sign of corresponding stresses,

\[
\begin{align*}
a_{11} &= \begin{cases} a_{11}^t = 1/E^t (\sigma_{\alpha} > 0), \\
a_{11}^c = 1/E^c (\sigma_{\alpha} < 0), \end{cases} \\
a_{22} &= \begin{cases} a_{22}^t = 1/E^t (\sigma_{\beta} > 0), \\
a_{22}^c = 1/E^c (\sigma_{\beta} < 0), \end{cases} \\
a_{33} &= \begin{cases} a_{33}^t = 1/E^t (\sigma_{\gamma} > 0), \\
a_{33}^c = 1/E^c (\sigma_{\gamma} < 0). \end{cases}
\end{align*}
\]

(2)

According to the symmetric nature of the compliance, off-diagonal compliances \( a_{ij} (i \neq j) \) are simplified to

\[
a_{12} (= a_{23} = a_{31}) = -\nu^t/E^t = -\nu^c/E^c = \text{constant.} \quad (3)
\]

These are not affected by the sign of stress.

In the first approximation of plate bending problems, a transverse normal stress and a transverse shear deformation are neglected. It is verified that if a plate is thin enough such an assumption gives sufficiently sound results. However, this assumption may not be employed for the moderately thick plate under consideration. Since analyses of plates with an arbitrary profile become very complicated, the following derivation of the governing equations is restricted to the cylindrical bending of a flat plate. The effect of transverse normal stress on the deformation of a plate is generally smaller than that of transverse shear, so we ignore the former in comparison with the latter.

As shown in fig. 2 a rectangular coordinate system \( x, y, z \) is set up as: \( y \), parallel to the generator of a plate; \( z \), perpendicular to the plane of a plate. Eqs (1) are transformed into the equations for cylindrical bending with respect to the \( x, y, z \) coordinate system as follows:

\[
\begin{align*}
\varepsilon_{x} &= a_{11}\sigma_{x} + a_{12}\sigma_{y} + (a_{33} - a_{11})n_{11}^{2}\sigma_{\gamma}, \\
0 &= a_{22}\sigma_{y} + a_{12}\sigma_{x}, \\
\varepsilon_{z} &= a_{12}(\sigma_{x} + \sigma_{y}) + (a_{33} - a_{11})(1 - n_{11}^{2})\sigma_{\gamma}, \\
\varepsilon_{xz} &= (a_{11} + a_{33} - 2a_{12})\tau_{xz},
\end{align*}
\]

(4)

where the last equation is expressed under the assumption that a similar relation of the conventional elastic material between shearing stress and strain would hold in an approximate manner because of the secondary nature of the transverse shear deformation effect on the total deformation when compared with that of in-plane stresses. Since the \( y \) direction corresponds to one of the principal directions from a geometrical consideration, we indicate it as parallel to the \( \beta \) direction. Principal directions in the \( xz \) plane are denoted by \( \alpha \) and \( \gamma \), respectively. In eqs (4), \( n_{11} \) stands for a cosine of an angle between \( \gamma \) and \( x \) axes.

Since the transverse normal stress \( \sigma_{z} \) is neglected, principal stresses in the \( xz \) plane are expressed as follows:

\[
\begin{pmatrix}
\sigma_{1} \\
\sigma_{2}
\end{pmatrix} = \frac{1}{2}\left(\sigma_{x} \mp (\sigma_{x}^{2} + 4\tau_{xz}^{2})^{1/2}\right).
\]

(5)
These equations both confirm the inequalities
\[ \sigma_1 \geq 0 \quad \text{and} \quad \sigma_2 \leq 0. \]  
(6)

Therefore, if we take either combination
\[ \sigma_1 = \sigma_\alpha, \quad \sigma_2 = \sigma_\gamma \quad \text{or} \quad \sigma_1 = \sigma_\gamma, \quad \sigma_2 = \sigma_\alpha, \]
a sum \( a_{11} + a_{33} \) remains as a constant, \( 1/E^t + 1/E^c \), where \( a_{11} \) and \( a_{33} \) depend on the sign of \( \sigma_\alpha \) and \( \sigma_\gamma \), respectively. From this situation and consideration of eq. (3) according to the symmetric nature of the compliance, a coefficient \( a_{11} + a_{33} - 2a_{12} \) appearing on the right-hand side of eq. (4) may be construed as a constant which is not affected by the sign of any stress.

Here we apply a method from Ambartsumyan for anisotropic plates or shells in consideration of the effect of a transverse shear. First, by introducing three functions, \( f(z) \), \( X_1(x) \) and \( X_2(x) \), where \( f(z) \) is a function of the coordinate of the thickness direction \( z \) and vanishes on both surfaces of the plate, \( f(\pm h/2) = 0 \), we express \( \tau_{xz} \), in order to satisfy boundary conditions on \( \tau_{xz} \) on both surfaces, as follows:
\[ \tau_{xz} = X_1(x) + X_2(x)z/h + f(z)\varphi(x). \]  
(7)

Denoting the lateral deflection and in-plane displacement in the \( x \) direction as \( w \) and \( u(x, z) \), respectively, a transverse shear strain \( \epsilon_{xz} \) is expressed, according to its definition, as follows:
\[ \epsilon_{xz} = \partial u/\partial z + \partial w/\partial x. \]  
(8)

Substituting eqs (7) and (8) into eq. (4)\(_4\), we obtain the following:
\[ \partial u/\partial z = (a_{11} + a_{33} - 2a_{12}) \{X_1(x) + X_2(x)z/h + f(z)\varphi(x)\} - \partial w/\partial x. \]  
(9)

Now, if we suppose that a lateral displacement \( w \) does not vary in the thickness direction \( z \), i.e. \( w = w(x) \), and take into account that the coefficient \( a_{11} + a_{33} - 2a_{12} \) is constant, after integration with respect to \( z \), the above equation becomes
\[ u = (a_{11} + a_{33} - 2a_{12}) \left\{ X_1(x)z + X_2(x)z^2/2h \right. \]
\[ \left. + \varphi(x) \int_0^z f(z)dz \right\} - z\partial w/\partial x + u_0(x), \]  
(10)

where \( u_0(x) \) stands for the magnitude of \( u \) at the middle plane of the plate, \( z = 0 \), i.e. \( u_0(x) = u(x, 0) \).

From eq. (10), a normal strain \( \epsilon_x \) in the \( x \) direction is formulated as follows:
\[ \epsilon_x = \partial u/\partial x = (a_{11} + a_{33} - 2a_{12}) \left\{ \partial X_1/\partial x \right. \]
\[ \left. + \partial X_2/\partial x z^2/2h + \partial \varphi/\partial x \int_0^x f(z)dz \right\} \]
\[ - z^2 \partial^2 w/\partial x^2 + \partial u_0/\partial x. \]  
(11)

The last two terms involved in the right-hand side of eq. (11) are the same as for the expression based on the conventional plate theory of Kirchhoff–Love, and then the first term of eq. (11) together with the expression of \( \epsilon_{xz} \) corresponds to a correction to the elementary theory of a bimodulus plate. If we indicate \( f(z) \) by an appropriate function, the in-plane displacement and strain components are expressed in concrete form.

From eqs (4)\(_{1-3}\) we obtain
\[ \sigma_x = \frac{(\epsilon_x + \epsilon_z - (a_{33} - a_{11})\sigma_\gamma)}{(a_{11} + a_{12} - 2a_{12}^2/a_{22})} \]  
(12)
and also from eqs (1) and the invariance identity of a summation of three principal strains, we find

\[
\sigma_\gamma = \left\{ (a_{11}a_{22} + a_{12}a_{22} - 2a_{12}^2)\epsilon_\gamma + (a_{12}^2 - a_{12}a_{22}) \times (\epsilon_x + \epsilon_z) \right\} / \left[ \left( a_{11}a_{22}a_{33} + 2a_{12}^3 - a_{12}^2 \right) \times (a_{11} + a_{22} + a_{33}) \right].
\]

(13)

For convenience in calculation, the principal strains \( \epsilon_a \) and \( \epsilon_b \) are selected as minimum and maximum ones, respectively. Validity of such a selection is illustrated later. Thus

\[
\epsilon_\gamma = \frac{1}{4} \left[ \epsilon_x + \epsilon_z + (\epsilon_x - \epsilon_z)^2 + \epsilon_{xz}^2 \right]^{1/2}.
\]

(14)

Furthermore, we neglect \( \epsilon_z \) in eqs (12)-(14) and substitute eq. (14) into eqs (12) and (13), obtaining the following equations:

\[
\sigma_x = a_{22} \left[ \epsilon_x \left( \frac{1}{2} a_{11}a_{22}a_{11} + a_{33} \right) + \frac{1}{2} a_{12}a_{22}(a_{11} - a_{11}) - a_{12}^2(a_{11} + a_{22} + a_{33}) \right.
\]

\[
+ 2a_{12}^2 \right] - \frac{1}{2} (a_{33} - a_{11})(a_{11}a_{22} + a_{12}a_{22} - 2a_{12}^2) \times (\epsilon_x^2 + \epsilon_{xz}^2)^{1/2} / \left[ \left( a_{11}a_{22} + a_{12}a_{22} - a_{12}^2 \right) \times (a_{11}a_{22}a_{33} + 2a_{12}^3 - a_{12}^2) \right].
\]

(15)

\[
\sigma_z = \frac{\epsilon_x(a_{11}a_{22} - a_{12}a_{22}) + (a_{11}a_{22} + a_{12}a_{22} - 2a_{12}^2)}{2(a_{11}a_{22}a_{33} + 2a_{12}^3 - a_{12}^2)}
\]

\[
- a_{12}^2(a_{11} + a_{22} + a_{33}) \right] \times \left[ \frac{1}{2} (a_{11} + a_{22} + a_{33}) \right].
\]

(16)

Then, elastic compliances \( a_{11} \) and \( a_{33} \) are determined by the following relations:

\[
a_{33} = \begin{cases} a_{33}^* (\sigma_\gamma > 0) \\ a_{33}^* (\sigma_\gamma < 0), \end{cases}
\]

(17)

\[
a_{11} = 1/\varepsilon + 1/\varepsilon^* - a_{33}^* \quad \text{(since } a_{11} + a_{33} = 1/\varepsilon + 1/\varepsilon^*). \quad \text{(18)}
\]

Also, since \( \sigma_\beta \) is expressed from eq. (4) as

\[
\sigma_\beta = \sigma_y = -a_{12}a_{22} \sigma_x,
\]

the value of \( a_{22} \) is distinguished by the sign of \( \sigma_x \) (if Poisson's ratios are assumed to be only positive)

\[
a_{22} = \begin{cases} a_{22}^* (\sigma_x > 0) \\ a_{22}^* (\sigma_x < 0). \end{cases}
\]

(19)

The strain energy per unit volume accumulated in a deformed body is

\[
V_0 = \frac{1}{2}(a_{11} \epsilon_x + \tau_{xz} \epsilon_{xz}).
\]

(20)

Substitution from eqs (15), (11), (7) and (4) into eq. (20) and integration of \( V_0 \) over a whole volume of the plate yields an expression of its total strain energy \( V \).

3. Method of analysis

One reason why stress or deformation analysis of the bimodulus materials encounters difficulty is that since the elastic constants in their constitutive or governing equations have direct dependence on the state of stress, i.e. the sign of principal stress, their proper value cannot be indicated at each material element in a deformed body \( \text{a priori} \). Of course, this obstacle cannot be overcome by a direct application of the conventional energy methods, which are now the most familiar and effective tool for structural analyses. Therefore, we here intend to follow a successive approximation numerical method based on the modified energy consideration of a mechanical system of the deformable body under study \[10\], starting from a close estimation of the desired solution.

Now, we introduce the following dimensionless values:

\[
X = x/a, \quad Z = z/h, \quad W = wE_0/\alpha h, \quad U = uE_0a/\alpha h^2;
\]

\[
\bar{\sigma}_x = \epsilon_xE_0a^2/\alpha h^2, \ldots, \quad \bar{\sigma}_x = \epsilon_xE_0a^2/\alpha h^2, \ldots, \quad (21)
\]

\[
A_{11} = a_{11}^*E_0, \ldots, \quad \Phi = a^2/\alpha h^2.
\]

where \( E_0 \) denotes a representative modulus of elasticity. A total potential energy, per unit width in the direction along a generator (y direction) of the deformed plate subjected to a uniformly distributed lateral load \( p \) is as follows:

\[
\pi = \frac{1}{4} \int_0^{h/2} \int_{-h/2}^{h/2} (\alpha_x \epsilon_x + \tau_{xz} \epsilon_{xz}) \, dz \, dx - \int_0^a p \, w \, dx
\]

\[
= \frac{p^2}{E_0} \alpha h(a/h)^4 \left[ \left( \frac{1}{2} - \frac{1}{2} \right) \int_0^{1/2} (\bar{\sigma}_x \bar{\epsilon}_x + \bar{\tau}_{xz} \bar{\epsilon}_{xz}) \, dl \right]
\]

\[
- (a/h)^4 \int_0^1 \bar{W} \, dX \right].
\]

(22)

A solution of the problem is obtained by minimization of \( \pi \), and a procedure for this is illustrated below.
First, we divide the plate into some finite partitions and approximate volume and surface integrations involved in eq. (22) by respective summation for each element, i.e. the sum of strain energy at the centroid of each element multiplied by its volume or area [10]. Then, since every strain energy of a plate element is calculated at ‘each element’, we may indicate εx or εy as a maximum or minimum principal strain at each element ‘independently’ of neighbouring ones. Therefore we may select, for instance, εx as the ‘maximum’ principal strain component as shown in eq. (14). Such a selection makes analysis easier especially in numerical calculation. However, if a whole plate is treated in a common field, εx and εy must be regarded as principal strains which vary continuously in the body, respectively.

Next, we assume displacement, u, w and the function φ in series form which satisfy prescribed boundary conditions. Coefficients involved in the series expression are determined so as to minimize the total potential energy π. As mentioned above, since the elastic compliance of each element in a deformed plate cannot usually be estimated beforehand, attention is initially paid to the material with identical Young's moduli in tension and compression, E' = E, which is conventional elastic material whose solution is obtained relatively easily by the usual methods. We take a ratio of the tensile modulus to the compressive one, E'/E, as a measure of bimodulus property. Then, the solution for the material with E'/E = 1 may be assumed as known, and thus it may be employed as an initial estimate for a solution to the material with E'/E slightly different from unity, i.e. E'/E = 1 + Δ(E'/E) where Δ(E'/E) denotes a small increment of the parameter E'/E. This successive procedure may be generalized to obtain a solution for the material with E'/E + Δ(E'/E), starting calculation from the known solution for the material with E'/E as a close approximation of the former. Such a stepwise approach makes it possible to obtain a solution for the material with any prescribed value of E'/E. A similar method of solution may be applied by lowering E'/E, for the material with E'/E ≤ 1 starting also from E'/E = 1.

4. Numerical example

As an illustration of the above equations governing plate bending and involving the effect of transverse shear deformation and a method of solution, a plate is considered which is infinitely long in the y direction and subjected to a uniformly distributed lateral load while being immovably supported along both edges, x = a and x = 0. In this case, both functions X1(x) and X2(x) from eq. (7) vanish from the boundary conditions on τxz on the top and bottom surfaces.

For simplicity’s sake we assume lateral and in-plane displacements, w, u0, as follows:

$$w = s_1 \sin \left( \frac{\pi x}{a} \right),$$  \hspace{1cm} (23)

$$u_0 = s_2 \sin \left( \frac{2\pi x}{a} \right) + s_3 \sin \left( \frac{4\pi x}{a} \right),$$  \hspace{1cm} (24)

where s1, s2 and s3 are unknown coefficients which must be determined as a solution. The function φ(x) appearing in eq. (7) may be estimated distinctly by the following consideration. A shear force Qx per unit length of the plate is defined by the relation

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} \, dz.$$  \hspace{1cm} (25)

The equilibrium equations of the plate are

$$dQ_x/dx + p = 0, \quad dM_x/dx - Q_x = 0.$$  \hspace{1cm} (26)

Substituting τxz of eq. (7) into eq. (25), it becomes

$$\int_{-h/2}^{h/2} \frac{h^2}{2} \int_{-h/2}^{h/2} \frac{h^2}{2} f(z) \phi(x) \, dz = \phi(x) \int_{-h/2}^{h/2} f(z) \, dz.$$  \hspace{1cm} (27)

And also substituting these into eq. (26), we obtain

$$d\phi/dx \int_{-h/2}^{h/2} f(z) \, dz + p = 0.$$  \hspace{1cm} (28)

Integrating eq. (28) with respect to x yields

$$\phi = \phi_0 - px \int_{-h/2}^{h/2} f(z) \, dz,$$  \hspace{1cm} (29)

where φ0 denotes a value of φ at x = 0. According to a condition of symmetrical deformation of the plate with respect to x = a/2,

$$Q_x = dM_x/dx = 0 \quad \text{at} \quad x = a/2.$$  

Thus, φ0 and therefore φ are determined as follows:

$$\phi_0 = pa/2 \int_{-h/2}^{h/2} f(z) \, dz, \quad \phi = p \left( \frac{a}{2} - x \right) \int_{-h/2}^{h/2} f(z) \, dz.$$  \hspace{1cm} (30)
The Ambartsumyan function \( f(z) \), which indicates a distribution of the transverse shear stress in the thickness direction, must be assumed so as to be as close as possible an approximation of the actual stress distribution. However, since detailed information on the stress state in the plate is not always known beforehand, and as has been reported for anisotropic plates [9] the choice of \( f(z) \) cannot introduce inadmissible error on the result, we here adopt the simplest distribution function, i.e. a quadratic parabolic function

\[
f(z) = \left( \frac{h}{2} \right)^2 - z^2, \tag{31}\]

which is known as the distribution function of the transverse shear stress in the elastic beam. Then, from eq. (30) \( \varphi(x) \) is expressed as

\[
\varphi(x) = \frac{6p}{h^3} \left( \frac{a}{2} - x \right). \tag{32}\]

Also, from eqs (10) and (11) together with eq. (32), the in-plane displacement \( u \) and the normal strain \( \varepsilon_x \) become

\[
u = \left( \frac{pz}{h^3} \right) \left( \frac{3}{2}h^2 - 2z^2 \right) \left( \frac{a}{2} - x \right) (a_{11} + a_{33} - 2a_{12})
- z \frac{dw}{dx} + u_0, \tag{33}\]

\[
\varepsilon_x = \left( - \frac{pz}{h^3} \right) \left( \frac{3}{2}h^2 - 2z^2 \right) (a_{11} + a_{33} - 2a_{12})
- z \frac{d^2w}{dx^2} + \frac{du_0}{dx}. \tag{34}\]

Numerical calculations were carried out for plates with a geometry of \( a/h = 4 \) and 8, and their deflections \( s_1 \) at \( x = a/2 \) are shown in figs 3 and 4, respectively. Poisson's ratio in compression is fixed throughout as \( \nu^c = 0.2 \), and a region of \( E^t/E^c \) is taken as \( \frac{1}{2} \leq E^t/E^c \leq 2 \). The tensile /Poisson's ratio is indicated by the relation \( \nu^t = \nu^c E^t/E^c \) (cf. eq. (3)).

A measure of bimodulus property \( E^t/E^c \) and its inverse \( E^c/E^t \) are taken for \( E^t/E^c \geq 1 \) and \( E^i/E^c \leq 1 \), respectively, as abscissae in figs 3 and 4. A solid line corresponds to the result based on the present method including the transverse shear deformation, and a chain line is obtained by setting \( \varphi = 0 \), i.e. \( \tau_{xz} = 0 \) in eq. (7); which means that the transverse shear deformation effect is disregarded. The fact that the chain line always runs below the solid one in both plates at \( a/h = 4 \) and 8 parallels the known trend of the deflection in the conventional elastic beams with or without consideration of transverse shear. The larger a parameter \( E^t/E^c \), the smaller the deflection of plate, and a relationship between solid and chain lines seems almost unchanged. Therefore, the larger \( E^t/E^c \), the more striking the effect of the transverse shear deformation on the deflection. For instance, when \( a/h = 4 \), the resulting discrepancy for \( \varphi = 0 \) in relation to \( \varphi \neq 0 \) is about 40% for \( E^t/E^c = 2 \) and 12% for \( E^c/E^t = 2 \), respectively. A comparison of results shown in figs 2 and 4 displays a more intensive effect of the transverse shear for the thicker plate of \( a/h = 4 \) than \( a/h = 8 \).

As a whole, it is worth noting that the transverse shear effect is remarkable in the thick bimodulus plate.
as well as the conventional elastic plate. Furthermore, we should pay attention especially to the effect in plates of material with a large $E^t/E^c$ ratio.

5. Concluding remarks

A method of analysis of plate bending was proposed, including the effect of transverse shear deformation in the bimodulus plate, by employing Ambartsumyan's line of thought for a weak moment shell. Derivation of the governing equations and a method of calculation were explained by an example problem of cylindrical bending of a moderately thick plate. Results of numerical calculation displayed pronounced effects of the transverse shear and difference of both moduli. Since this paper mainly stressed a method to introduce the transverse shear deformation effect into an analysis of bimodulus plate bending, only a simple example was employed. It also must be borne in mind that the present results and discussion are restricted to assumed approximate displacement functions expressed as eqs (23) and (24). More rigorous and advanced analysis is possible if, for instance, one uses displacement functions expanded in series form with many more terms.

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Nomenclature

\[ Q_x \] = shear force per unit length
\[ u, u_0 \] = in-plane displacement
\[ V, V_0 \] = strain energy
\[ w \] = deflection of plate
\[ x, y, z \] = rectangular coordinate system
\( ( \ )^t \) = tension value
\( ( \ )^c \) = compression value
\[ \alpha, \beta, \gamma \] = principal stress coordinate system
\[ e \] = strain component
\[ \nu \] = Poisson's ratio
\[ \pi \] = potential energy
\[ \sigma, \tau \] = stress component.

References