LARGE DEFLECTION ANALYSIS OF BIMODULUS
ANNULAR AND CIRCULAR PLATES USING
FINITE ELEMENTS

R. S. SRINIVASAN and L. S. RAMACHANDRA
Department of Applied Mechanics, Indian Institute of Technology, Madras 600 036, India
(Received 1 February 1988)

Abstract—In the present paper, the finite element method is used to solve the large-deflection bending problem of annular and circular bimodulus plates. The material model suggested by Jones is adopted to represent the bilinear stress-strain relations of bimodulus material. The resulting nonlinear equations are solved, using the Newton–Raphson iterative procedure. In the case of bimodulus composite structures, at any point, the position of neutral axis is a function of the state of stress at that point and is evaluated following iterative procedures. Numerical work has been done for two different types of boundary conditions, viz. clamped and simply supported cases. The results have been compared for particular cases and they are presented in the form of graphs and tables.

INTRODUCTION
In the case of plates carrying heavy loads and whose deflections are of the same order as that of the thickness of plate, a more rigorous analysis is called for. In such cases, nonlinear plate theory which takes into account the coupling of membrane and bending stresses is applied. The geometric nonlinearity is incorporated in a strain–displacement relation by taking the large rotation term into account. In the present case, circular and annular plates with various boundary conditions are analyzed. They have a wide application to various fields (e.g. in pressure vessels, liquid storage tanks etc.).

Composite materials are gaining importance in the field of aeronautical, mechanical and marine engineering. The main reason for their use is that of high strength to weight ratio. Many of the composites have different stress–strain curves in compression and tension. Such composites are termed as bimodulus materials. In the case of bimodulus composites, the constitutive matrix is a function of stress. Hence the analysis of structures made up from bimodulus materials is more involved. The stress–strain relation is actually curvilinear, but it is approximated as bilinear with different slopes. To simplify the analysis, different mathematical models for the stress–strain relationship of the material have been proposed (see [1] and [2]). A brief review of all the models has been given by Tabaddor [3].

In the literature, one comes across very few papers which deal with the nonlinear analysis of bimodulus composite plates. Kamiya [4] has dealt with the problem of large deflection analysis. He has analyzed a clamped circular plate. The finite difference method is adopted to solve the governing differential equations using iterative procedures.

He has used the Ambartsumyan model in which $a_0$ (the elements of the compliance matrix) is assumed to be symmetric. This implies $a_{11} (= a_{22})$ is constant and is not a function of the state of stress at any point.

In the present work the finite element technique adopted for linear analysis [5] has been extended for the large deflection analysis of bimodulus circular and annular plates subjected to axisymmetric lateral load. Annular finite elements have been used to discretize the structure. To the best of the authors' knowledge no work has been done on the large deflection analysis of bimodulus annular plates. Two different types of boundary conditions are considered for annular plates. In the case of a circular plate, a very small central hole is left with free boundary conditions. The governing equilibrium equation is formulated on the basis of von Karman's assumptions of a large deflection of the plate and by the method of minimizing potential energy. The nonlinear equations are written according to the standard nomenclature introduced by Rajasekaran and Murray [6]. The modified Newton–Raphson method is used to solve the nonlinear equations. The numerical results are presented in the form of graphs.

FORMULATION
In the present work the Jones [1] Weighted Compliance matrix model is used to describe the mechanical behaviour of composite materials. The strain–stress relation for the plane stress condition is

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix}.$$  

(1)
The compliances $a_{ij}$ are related to the elastic moduli and Poisson's ratio as follows.

From eqn (1) one obtains

$$\begin{bmatrix}
\sigma_r \\
\sigma_0
\end{bmatrix} =
\begin{bmatrix}
E_{11} & E_{12} \\
E_{12} & E_{22}
\end{bmatrix}
\begin{bmatrix}
\epsilon_r \\
\epsilon_0
\end{bmatrix}
$$

where $[E]$ is the stiffness matrix corresponding to the plane stress condition.

Strain energy of the annular element for axisymmetric case (see Fig. 1) is given by the expression

$$U = \frac{1}{2} \int_V \left\{ \sigma \right\}^T [E] \{\epsilon\} \, dV$$

According to Von Karman's strain-displacement relation for large deflections of the middle surface,

$$\epsilon_r = \frac{u}{r} + \frac{1}{2} \left( w' \right)^2$$

$$\epsilon_0 = u/r,$$
in which $u$ represents the displacement along $r$-direction at a point and $u_0$ denotes displacement of the middle plane. Then

$$
\epsilon = u_0 - Z w'' + \frac{1}{2} (w')^2
$$

$$
\epsilon_\theta = \frac{u_0}{r} - \frac{Z}{r} w'.
$$

(7)

In further calculations the subscript '0' is omitted.

Equation (5) can be written as

$$
U = U_L + U_{NL}
$$

where

$$
U_L = \frac{1}{2} \int \left\{ A_{11} (u')^2 + A_{12} \frac{u^2}{r^2} + 2 A_{12} \frac{u}{r} u' \right\} \, dA
$$

$$
+ \left[ D_{11} (w'')^2 + D_{12} \frac{1}{r^2} (w')^2 + 2 D_{12} \frac{1}{r} w'' w' \right] \, dA
$$

$$
- \left[ 2 B_{11} u' w'' + 2 B_{12} \frac{u}{r} w'' + 2 B_{12} \frac{1}{r} u' w' \right. 

$$

$$
+ 2 B_{12} \frac{u}{r} w'' \right\} \, dA
$$

(9)

and

$$
U_{NL} = \frac{1}{2} \int \left\{ A_{11} (w')^2 + A_{11} u' (w')^2 - B_{11} w'' (w')^2 

+ A_{12} \frac{u}{r} (w')^2 - B_{12} \frac{1}{r} (w')^2 w' \right\} \, dA
$$

(10)

in which the stretching, stretching-bending and bending stiffnesses are defined as

$$
[A_0 B_0 D_0]^T = \int_{-h/2}^{h/2} E_A [1 Z \ Z^2] \, dZ
$$

$$
i, j = 1, 2.
$$

(11)

For finite-element approximations, the following displacement fields are assumed:

$$
u = c_1 + c_2 r + c_3 r^2 + c_4 r^3
$$

$$
w = c_1 + c_2 r + c_3 r^2 + c_4 r^3 + c_5 r^4 + c_6 r^5.
$$

(12)

Let $[\delta] = [T][c]$ where

$$
{\delta}^T = [(u u_0 w w_0), \ (u u_0 w w_0)]
$$

$$
{C}^T = [c_1 \ c_2 \ c_3 \ c_4 \ c_5 \ c_6 \ c_7 \ c_8 \ c_9 \ c_{10}].
$$

(13)

Then

$$
\{\delta\} = [T]^{-1}[\delta] = [TIN][\delta].
$$

Let

$$
\chi = |F|\{c\},
$$

(14)

and

$$
\{\chi\}^T = \begin{bmatrix} u & \frac{du}{dr} & \frac{d^3u}{dr^3} & \frac{dw}{dr} \end{bmatrix}.
$$

Using eqn (12), one can write the expressions (9) and (10) as follows:

$$
U_L = \frac{1}{2} 2 \pi h \{\delta\}^T [TIN]^T \int [F]^T [E] [F] \, dr [TIN] \{\delta\}
$$

$$
= \frac{1}{2} \{\delta\}^T [K_L] \{\delta\}
$$

(13)

$$
U_{NL} = \frac{1}{2} \{\delta\}^T 2 \pi h [N_L] \{\delta\}
$$

$$
+ \frac{1}{2} \{\delta\}^T 2 \pi h [N_L] \{\delta\}
$$

(14)

$$
= \{\delta\}^T \left[ \frac{1}{2} [N_L] + \frac{1}{2} [N_L] \right] \{\delta\}
$$

where

$$
[N]_i = 2 \pi h
$$

$$
\begin{bmatrix}
0 & 0 & 0 & A_{12} \frac{u'}{r} & 0 \\
0 & 0 & 0 & A_{11} u' & 0 \\
0 & 0 & 0 & 0 & 0 \\
A_{12} w' & A_{11} w' & 0 & A_{11} w' + A_{12} u' & -B_{11} w' \\
-B_{11} w' & -B_{11} w' & 0 & 0 & 0 \\
\end{bmatrix}
$$

(15)
The incremental stiffness matrix is
\[
[K] = [K]_L + [N]_1 + [N]_2.
\]

The external work done by the lateral load \( q \) is given by
\[
W = \int_0^{\gamma} \{ w \}^T q r dr d\theta. \tag{17}
\]

The displacement function \( w \) may be expressed as:
\[
\{ w \} = [0 0 0 1 r^2 r^4 r^6] \{ c \} \tag{18}
\]
or
\[
\{ w \} = \{ \phi \}^T \{ c \}.
\]
Substituting for \( \{ c \} \), one obtains
\[
\{ w \} = \{ \phi \}^T [TIN] \{ \delta \}.
\]
Now, eqn (17) takes the following form.
\[
W = 2\pi \{ \delta \}^T [TIN]^T q \int_{r_1}^{r_2} \{ \phi \} r dr. \tag{19}
\]

The load vector \( Q \) for an element, may be written as
\[
Q = 2\pi [TIN]^T q \int_{r_1}^{r_2} \{ \phi \} r dr. \tag{20}
\]

**SOLUTION**

In the case of bimodulus materials, the evaluation of the stiffness matrix poses a problem, as the position of the neutral axis is not known \textit{a priori}, whereas at any point it depends upon the state of stress and the curvature at that point. In any direction (say \( r \)) depending upon the nature of the stress distribution and position of neutral axis, it is possible to have six cases of strain distribution (see Fig. 2). In the case of two-dimensional problems, the neutral axes along \( r \) and \( \theta \) directions have to be determined. Hence along normal to mid surface the state of stress distribution will pertain to any one of the 36 cases of strain distribution. In any particular case, there will be different zones of strain conditions over the thickness of the plate and the appropriate constitutive matrices are adopted for the calculation of stiffness matrix. The neutral axes are located by iterative processes. For each load, the iterations are carried out until:
\[
\left| \frac{\eta_{n+1} - \eta_n}{\eta_n} \right| < 0.1 \times 10^{-3},
\]
where \( \eta \) is the depth of the neutral axis from the mid thickness and \( n \), the number of iterations.

---

Fig. 2. Different combinations of strain distribution in \( r \) and \( \theta \) directions.
Large deflection analysis of bimodulus plates

The resulting nonlinear algebraic equations of the plate are solved using the modified Newton–Raphson method. In this procedure, the load is applied in small steps, and the structure is assumed to behave linearly over a given step-size. Initially the equilibrium equation of the structure is not satisfied, as the stiffness matrix is approximate over a given load step. So the increments in nodal displacements are calculated for the unbalanced loads and are added to the global displacement vector.

NUMERICAL WORK AND DISCUSSION

A convergence study for a bimodulus annular plate with the following material properties has been done.

\[ E_1 = 0.5 \times 10^7 \text{ psi} \]
\[ v_1 = 0.10 \]
\[ G_1 = 0.45 \times 10^7 \text{ psi} \]
\[ E_2 = 1.0 \times 10^7 \text{ psi} \]
\[ v_2 = 0.20 \]
\[ G_2 = 0.42 \times 10^7 \text{ psi} \]

The plate considered is fixed at the outer edge and free at the inner edge. The results obtained from a uniformly distributed lateral load are presented in Table 1, for various loads. The convergence of the deflection at the free edge and curvature at the outer edge has been studied by considering 4, 8, 12 and 16 elements. There is good convergence of solutions and for further numerical work, 12 elements have been considered.

To verify the formulation, isotropic circular plates with clamped and simply supported (\( u = 0 \) at support) boundary conditions were considered. The plates are subjected to a uniformly distributed lateral load. The results for deflections and bending moments have been compared (see Figs 3 and 4) with those given in [7] using the Dynamic Relaxation method. The results are found to agree well. Similarly the values for annular plates with the outer edge (i) clamped and (ii) simply supported, the inner edge being free for both the cases, have also been compared in Figs 5 and 6. The results compare well.

The bimodulus clamped circular plate is analysed for two cases of \( E_1/E_2 = \frac{1}{3} \) and 2 and the results are compared with those of Kamiya in Figs 7 and 8. The results for deflection and bending moment agree well. Whereas in both cases the in-plane displacement does not agree well, however it may be noted here that the in-plane displacements are of very small order when compared to the out-of-plane displacement. Kamiya has used the Ambartsumyan material model in his analysis. For the material which satisfies the relation, viz. \( v_1/E_1 = v_2/E_2 \), the Jones model reduces to that of Ambartsumyan’s.

The numerical work for a bimodulus circular plate with simply supported boundary conditions has been done. The variation of deflections, moments and stress resultants with lateral load are shown in Fig. 9. The results for a bimodulus annular plate with clamped and simply supported boundary conditions for various \( (b/a) \) ratios have been obtained and they

| Table 1. Convergence study |

\[
\begin{array}{cccccc}
\text{Load parameter} & \frac{qa}{E_h} & \text{No. of Elements} \\
\text{No.} & 4 & 8 & 12 & 16 \\
1 & 0.3217 & 0.3225 & 0.3273 & 0.3257 \\
2 & 0.6201 & 0.6148 & 0.6253 & 0.6212 \\
3 & 0.8613 & 1.0815 & 0.8320 & 1.0725 \\
4 & 0.8487 & 1.0960 & 0.8487 & 1.0960 \\
5 & 1.2761 & 1.2672 & 1.2796 & 1.3034 \\
\end{array}
\]
Fig. 3. Variation of deflection at centre, moment and force at outer edge, for isotropic clamped circular plate.

Fig. 4. Variation of deflection at centre, moment and force at outer edge for isotropic simply supported circular plate.
Fig. 5. Variation of deflection at inner edge, moment and force for isotropic clamped annular plate.

Fig. 6. Variation of deflection at inner edge, moment and force for isotropic simply supported annular plate.
Fig. 7. Variation of displacement and moment for bimodulus circular clamped plate.

Fig. 8. Variation of displacement and moment for bimodulus circular clamped plate.
Fig. 9. Variation of deflection at centre, moment and force at outer edge for simply supported bimodulus circular plate.

Fig. 10. Variation of deflection at inner edge, moment and force for bimodulus clamped annular plate.
Fig. 11. Variation of deflection at inner edge, moment and force for bimodulus clamped annular plate.

Fig. 12. Variation of deflection at inner edge, moment and force for bimodulus simply supported annular plate.
Large deflection analysis of bimodulus plates are presented in Figs 10–13. It may be noted that the material property used for this study is the same as that used for the convergence study in Table 1.

In this paper, the nonlinear analysis of circular and annular bimodulus plates subjected to a lateral load has been performed using the finite element method. The numerical results for certain particular cases have been compared with values available in the literature, hence ensuring the correctness of the analysis and numerical work. Thus, the applicability of the finite element method to nonlinear analysis of bimodulus plates has been demonstrated.

Numerical results for certain cases have also been given in the hope that they may be useful to future researchers.

REFERENCES