Axisymmetric buckling and post-buckling of bimodulus annular plates

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The buckling and post-buckling analysis of bimodulus circular and annular plates is presented. Annular finite elements are used to solve the problem. In the case of bimodulus materials, the elastic constants in tension and compression are different. The constitutive matrix based on the 'Jones model' is used for the analysis. The post-buckling problem is solved as an eigen-value problem with deflexion at a point, as an independent variable. Numerical results are compared for particular cases.

Keywords: Buckling, post-buckling, circular plates, annular plates, bimodulus materials
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where subscripts ‘t’ and ‘c’ denote tension and compression respectively.

if \( \sigma_t < 0 \) and \( \sigma_c < 0 \),
\[
\begin{align*}
\alpha_{11} &= 1/E_c, & \alpha_{12} &= -v_c/E_c, & \alpha_{22} &= 1/E_c \\
\alpha_3 &= 1/E_t, & \alpha_{45} &= -v_t/E_t, & \alpha_{55} &= 1/E_t
\end{align*}
\]
if \( \sigma_t > 0 \) and \( \sigma_c < 0 \),
\[
\begin{align*}
\alpha_{11} &= 1/E_t, & \alpha_{12} &= -k_t v_t/E_t - k_c v_c/E_c, & \alpha_{22} &= 1/E_t \\
\alpha_3 &= 1/E_c, & \alpha_{45} &= -k_c v_c/E_c - k_t v_t/E_t, & \alpha_{55} &= 1/E_c
\end{align*}
\]
where,
\[
k_t = \frac{|\sigma_t|}{|\sigma_t| + |\sigma_c|} \quad \text{and} \quad k_c = \frac{|\sigma_c|}{|\sigma_t| + |\sigma_c|}
\]

From equations (1) and (2):

\[
\{ \sigma \} = [E] \{ \varepsilon \}
\]

where \([E]\) is the stiffness matrix corresponding to the plane stress condition. The strain energy of an annular element may be written as:

\[
U = U_L + U_{NL}
\]
in which \(U_L\) and \(U_{NL}\) denote strain energy due to linear and nonlinear strains respectively. For the axisymmetric case, the strain displacement relations are given by:

\[
\begin{align*}
\varepsilon_r &= \frac{u}{r} - z \omega + \frac{1}{2} (w')^2 \\
\varepsilon_\theta &= \frac{r}{z} \omega
\end{align*}
\]

where \( \dot{\cdot} \) denotes differentiation with respect to \( r \). The strain energy expressions can then be written as:

\[
U_L = \frac{1}{2} \int_A \left[ \left( A_{11} u'^2 + A_{22} \left( \frac{u}{r} \right)^2 + 2 A_{12} \frac{u}{r} u' \right) w'' + D_{11} (w'')^2 + D_{22} \left( \frac{r}{z} \right)^2 (w')^2 + 2 D_{12} \frac{r}{z} w' w'' \\
- 2 B_{11} u' w'' - 2 B_{22} \frac{u}{r} w' + 2 B_{12} \frac{r}{z} u' w'' \\
+ 2 B_{22} \frac{u}{r} w'' \right] \, dA
\]

and

\[
U_{NL} = \frac{1}{2} \int_A \left[ \frac{1}{4} A_{11} (w')^4 + A_{11} u'(w')^2 - B_{11} (w')^2 \\
+ A_{12} \frac{u}{r} (w')^2 - B_{12} (w')^2 w' \right] \, dA
\]

where stretching, stretching-bending and bending stiffnesses are defined as:

\[
[A_{ij} B_{ij} D_{ij}]^T = \int E_i [1 z^2]^T \, dz
\]

where \( h \) is the thickness of the plate. The displacement field is expressed by the following two polynomials:

\[
\begin{align*}
 u &= c_1 + c_2 r^2 + c_3 r^3 + c_4 r^4 \\
 w &= c_5 + c_6 r + c_7 r^2 + c_8 r^3 + c_9 r^4 + c_{10} r^5
\end{align*}
\]

Using equation (10), equations (7) and (8) may be written as:

\[
U_L = \frac{1}{2} \{ \delta \}^T [K]_L \{ \delta \}
\]
in which \{ \delta \} = \{ [u u_\theta w w_\theta] \} where the subscripts 1 and 2 refer to the edges \( r = r_1 \) and \( r_2 \) respectively, and

\[
U_{NL} = \frac{1}{2} \{ \delta \}^T \{ [K]_c + 1/2 [K]_t \} \{ \delta \}
\]

Work done by the edge force (N/unit length) is given by:

\[
W = \frac{1}{2} \int_{r_1}^{r_2} N \left( \frac{dW}{dr} \right) r \, dr
\]

The radial force \( N_r \) at any point in the plate (as shown in the inset in Figure 2) is given by:

\[
N_r = \frac{-Na^2}{(a^2 - b^2)} \left( 1 - \frac{r^2}{a^2} \right)
\]

Combining equations (10) and (13) gives:

\[
W = \frac{1}{2} \{ \delta \}^T \lambda [G] \{ \delta \}
\]

in which \( \lambda = Na^2/(a^2 - b^2) \). The total potential energy of the system is:

\[
V = \frac{1}{2} \{ \delta \}^T [K] \{ \delta \} + \frac{1}{2} \{ \delta \}^T [G] \{ \delta \}
\]

where \([K] = \{ [K]_L + 1/2 [K]_c + 1/4 [K]_t \}

At the buckling load, \( V \) is minimum and hence

\[
[K]_L + \lambda [G] \{ \delta \} = \{ 0 \}
\]

The lowest eigen-value of this equation corresponds to the buckling load of the system.

In the case of post-buckling analysis, the governing equation takes the form:

\[
[K]_L + \lambda [G] \{ \delta \} = \{ 0 \}
\]

Method of application

In the case of bimodulus materials the constitutive matrix is not constant. It is a function of stress at any given point. Also, for bimodulus materials it is possible to have six cases of strain distribution (as shown in Figure 1) for any direction. Hence, at any point inside the domain of a two-dimensional body, the state of stress pertains to any one of the 36 (i.e. 6 \times 6) cases of strain distribution. Therefore it is possible to have different zones of stress conditions over the thickness of the plate and the appropriate constitutive matrices are adopted to calculate the stiffness matrix.

In the case of linear buckling, the constitutive relationships as they stand in the prebuckling state is used. Since the entire plate is in compression in both \( r \) and \( \theta \) directions, the stress–strain relationship corresponding to \( \sigma_t < 0 \) and \( \sigma_c < 0 \) has been adopted to calculate the stiffness matrix. By solving the eigen-value problem (equation 17) the critical load is obtained.

For the post-buckling analysis a different procedure is adopted. Since the position of the neutral axes is not
known a priori, the strain distribution and the neutral axes positions in the \( r \) and \( \theta \) directions are assumed for the first iteration. Based on these values, the stiffness matrix is calculated and by solving equation (18), \( \{\delta\} \) can be found. From this, new values of strains and neutral axes positions are determined. The above procedure is repeated until the appropriate convergence criterion is satisfied.

To trace the post-buckling behaviour, the \( i \)th component of the displacement vector is treated as an independent variable. By incrementing this component, the stiffness matrix is calculated. Solving equation (18), the eigen-value (\( \lambda \)) and the eigen-vector are determined. Using this eigen-vector, the stiffness matrix is updated and the load is evaluated, holding the \( i \)th component of the displacement vector to a constant value. This procedure is repeated to obtain the converged eigen-value (which gives the load), satisfying the prescribed criterion, i.e., the difference between successive eigen-values is limited to \( 1000^{-3} \) of the eigen-value.

### Numerical examples

For the convergence study, a bimodulus annular plate with outer edge fixed and inner edge free is considered. The properties of the bimodulus material used are:

\[
\begin{align*}
E_t &= 34.474 \text{ GPa}(0.5 \times 10^7 \text{ psi}), \quad v_t = 0.10, \\
G_t &= 31.0266 \text{ GPa}(0.45 \times 10^7 \text{ psi}) \\
E_c &= 68.948 \text{ GPa}(0.1 \times 10^7 \text{ psi}), \quad v_c = 0.20, \\
G_c &= 28.9581 \text{ GPa}(0.42 \times 10^7 \text{ psi})
\end{align*}
\]

The results are presented in Table 1. It may be noted that the solution converges well. For further study a mesh with 6 elements is considered.

To verify the formulation, buckling loads for isotropic annular plates were determined and compared with the values given in Reference (6). They are presented in Table 2 and the results are found to agree well. The values of the buckling coefficient \( \kappa(=N_{cr}a^2/D) \) for clamped-free and simply supported-free annular plates for different \( b/a \) ratios are presented in Table 3 along with the values given by Juang and Chen* for the case of \( E_t/E_c = 2.0 \). The ratio of radius to thickness is taken as 10 and \( v_{c} = 0.20 \). It can be observed from the table that there is some discrepancy between the authors' values and those of Juang and Chen. It may be noted that, in Reference (4) the neutral axis is inside the plate. This is possible only when there are finite deflexions; otherwise, the neutral axis will be outside the plate since the entire plate is in compression and there will be infinitesimally small lateral deflexions at the time of bifurcation. Moreover, their work is for a thick plate taking shear deformation into account and they have approximated the depth of neutral axes as \( Z = \frac{1}{2}(Z_t + Z_c) \). Hence it is likely that the discrepancy in the results may be due to the above factors.

### Table 1: Convergence study for the linear buckling

<table>
<thead>
<tr>
<th>No. of elements</th>
<th>Buckling coefficient, ( \kappa )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>40.134</td>
</tr>
<tr>
<td>6</td>
<td>40.134</td>
</tr>
<tr>
<td>8</td>
<td>40.134</td>
</tr>
</tbody>
</table>

\( N_{cr} = \frac{D_t}{\rho^2} \) where \( D_t = E_t h^3/12(1 - v_t^2); \ b/a = 0.4 \)

### Table 2: Comparison of results for isotropic annular plate with clamped outer edge and free inner edge

<table>
<thead>
<tr>
<th>( b/a )</th>
<th>Authors</th>
<th>Reference (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>14.682</td>
<td>14.682</td>
</tr>
<tr>
<td>0.1</td>
<td>13.741</td>
<td>13.848</td>
</tr>
<tr>
<td>0.2</td>
<td>13.289</td>
<td>13.393</td>
</tr>
<tr>
<td>0.3</td>
<td>14.694</td>
<td>14.691</td>
</tr>
</tbody>
</table>

\( v = 0.30 \)

### Table 3: Bimodulus annular plates

<table>
<thead>
<tr>
<th>( b/a )</th>
<th>Clamped-free*</th>
<th>Simply supported-free*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Authors</td>
<td>Reference (4)</td>
</tr>
<tr>
<td>0.4</td>
<td>8.51</td>
<td>11.18**</td>
</tr>
<tr>
<td>0.6</td>
<td>18.23</td>
<td>23.33</td>
</tr>
</tbody>
</table>

\( N_{cr} = \frac{D_t}{a^2}; \ inner \ edge \ is \ free; \ ** \ from \ graph \ in \ Reference \ (4) \)

The variation of the buckling coefficient \( \kappa \) vs \( b/a \) for clamped-free and simply supported-free bimodulus annular plates is shown in Figure 2. The material properties were the same as those used for the convergence study.

In the case of post-buckling analysis, a clamped circular plate is considered for the convergence study. The material properties were the same as those used for the case of linear buckling. The results for different meshes...
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Figure 2 Variation of buckling coefficient

Table 4 Convergence study for the post-buckling case

<table>
<thead>
<tr>
<th>Values of $N_{cr}^2/E_c h^2$</th>
<th>Number of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td>$w \backslash \mu$</td>
<td></td>
</tr>
<tr>
<td>0.20</td>
<td>0.901</td>
</tr>
<tr>
<td>0.40</td>
<td>0.914</td>
</tr>
<tr>
<td>0.60</td>
<td>0.939</td>
</tr>
<tr>
<td>0.80</td>
<td>1.027</td>
</tr>
<tr>
<td>1.0</td>
<td>1.104</td>
</tr>
<tr>
<td>1.20</td>
<td>1.193</td>
</tr>
<tr>
<td>1.40</td>
<td>1.318</td>
</tr>
<tr>
<td>1.60</td>
<td>1.442</td>
</tr>
</tbody>
</table>

* deflexion at the centre

are presented in Table 4. The convergence is found to be good. For further calculations a mesh with 12 elements was used. To check the validity of the formulation, the post-buckling analysis of isotropic clamped circular plate was carried out and the results are compared with values from the literature [7] in Table 5. The results are found to compare well.

The post-buckling path for the bimodulus clamped circular plate and clamped-free annular plate for different values of $\eta = E_t/E_c$ are shown in Figures 3 and 4. From these figures it can be observed that the stiffness of the plate increases as the value of $\eta$ increases.

References

1 Jones, R.M. 'Stress-strain relations for material with different moduli in tension and compression', AIAA J., 1977 15, 16-23
3 Jones, R.M. 'Buckling of stiffened multilayered circular cylindrical shells with different orthotropic moduli in tension and compression', AIAA J., 1971 9, 917-923
6 Huang, C.L. 'Post-buckling of an annulus', AIAA J., 1973 11, 1606-1612