BENDING OF BIMODULUS ANNULAR PLATES

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Abstract—Bending analysis of axisymmetric annular circular plates of bimodulus material subjected to uniform lateral load is dealt with in this paper. The finite element method is used to formulate the governing equations of equilibrium. An iterative technique is used to find the position of the neutral surface. Numerical work has been done for plates with different sizes and boundary conditions.

INTRODUCTION

Composite materials are receiving increased attention in structural applications due to their high strength to weight ratio. They are mainly used in aircraft and spacecraft structures which are more weight sensitive. Many composite materials exhibit different elastic properties in compression and tension. For such composites, the actual stress-strain relationship is approximated with a bilinear one which is a mathematical simplification of the physical phenomenon (Fig. 1). They are called bimodulus composite materials.

Timoshenko [1] has considered the pure bending problem of a beam made up of elastic material with different elastic moduli in tension and compression. Analysis of plates made up of bimodulus material began with the work of Ambartsumyan (vide [2]). Kamiya [3] studied the large deflection of circular clamped plates, subjected to uniform lateral load. He has adopted the "Ambartsumyan material model". The energy approach has been used to solve the problem. Jones and Morgan [4] have treated the cylindrical bending of cross-ply laminates. Bert et al. [5] have done the bending analysis of thin rectangular bimodulus plates. Their approach is through the governing differential equations. A finite element analysis of anisotropic bimodulus plate has been presented by Reddy and Chao [6].

In this paper annular circular isotropic bimodulus plate (Fig. 2) subjected to uniformly distributed lateral load has been analysed using finite element method, with axisymmetric annular circular element. The stiffness matrix has been developed on the lines followed in [7]. The material model proposed by Jones [8] is used here to formulate the problem.

The main difficulty in the analysis of bimodulus plates is that of locating the neutral surface. Since the element moduli depend on the sign of the stress which is unknown a priori, the iterative technique has been adopted to find the layer in which there is no strain.

The numerical results are presented for plates with clamped and simply supported boundary conditions.

FORMULATION

The material model [8] can be written as

\[
\begin{align*}
\epsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y \\
\epsilon_y &= a_{13}\sigma_x + a_{22}\sigma_y.
\end{align*}
\]

The compliances \(a_{11}, a_{12}\) and \(a_{22}\) will be found according to:

- If \(\sigma_x > 0\) and \(\sigma_y > 0\),
  \[
  a_{11} = 1/E_x, \quad a_{12} = -\nu_{xy}/E_x, \quad a_{22} = 1/E_y;
  \]
- If \(\sigma_x < 0\) and \(\sigma_y < 0\),
  \[
  a_{11} = 1/E_x, \quad a_{12} = -\nu_{xy}/E_y, \quad a_{22} = 1/E_y;
  \]
- If \(\sigma_x > 0\) and \(\sigma_y < 0\),
  \[
  a_{11} = 1/E_x, \quad a_{12} = k_x(a_{12}) + k_y(a_{12}), \quad a_{22} = 1/E_y;
  \]
- If \(\sigma_x < 0\) and \(\sigma_y > 0\),
  \[
  a_{11} = 1/E_y, \quad a_{12} = k_x(a_{12}) + k_y(a_{12})/E_y, \quad a_{22} = 1/E_y.
  \]

The values of \(k_x\) and \(k_y\) are given by the following expressions:

\[
\begin{align*}
k_x &= |\sigma_x|/(|\sigma_x| + |\sigma_y|) \\
k_y &= |\sigma_y|/(|\sigma_x| + |\sigma_y|).
\end{align*}
\]
The stresses can be expressed in terms of strains as:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
E_{11} & E_{12} \\
E_{21} & E_{22}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]  

(4)

or

\[
\{\sigma\} = [E]\{\epsilon\}
\]  

(5)

where \([E]\) is the stiffness matrix corresponding to the plane stress condition of the bimodulus material. The polar coordinates originate from the midplane of the plate. The Z-axis is normal to the mid plane \((r\theta\text{-plane})\) of the plate and is positive downwards.

Strain energy of the annular element (Fig. 3) is given by:

\[
U = \frac{1}{2} \int \int \left\{ \epsilon \right\}^T [E] \{\epsilon\} \, dV.
\]  

(6)

Substituting eqn (5) into (6) one gets

\[
U = \frac{1}{2} \int \int \{\epsilon\}^T [E] \{\epsilon\} \, r \, d\theta \, dz.
\]  

(7)

But

\[
\epsilon_r = \frac{1}{r} \frac{\partial u}{\partial r} - \frac{z}{r^2} \frac{\partial w}{\partial r}, \quad \epsilon_\theta = \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{z}{r^2} \frac{\partial w}{\partial \theta}.
\]  

(9)

Substituting the values of \(\epsilon_r, \epsilon_\theta\) in eqn (8) and simplifying,

\[
U = \frac{1}{2} \pi h \int \{x\}^T [E] \{x\} \, r \, dr.
\]  

(10)

where

\[
[E] = \begin{bmatrix}
[A] & [B] \\
[B]^T & [D]
\end{bmatrix}
\]

(11)

and

\[
\{X\}^T = \begin{bmatrix}
du & u^2 & w \frac{1}{r} & \frac{dw}{dr} & \frac{d^2w}{dr^2} & \frac{d^3w}{dr^3}
\end{bmatrix}^T.
\]  

(12)

FINITE ELEMENT FORMULATION

For inplane and lateral displacements, two separate polynomials are assumed as

\[
u = c_1 + c_2 r + c_3 r^2 + c_4 r^3
\]

\[
w = c_5 + c_6 r + c_7 r^2 + c_8 r^3 + c_{10} r^4.
\]  

(13)

From this it can be written as

\[
\{x\} = [F] \{c\}.
\]  

(14)

where

\[
[F] = \begin{bmatrix}
0 & 1 & 2 & 3 & r^2 \\
1 & r & 6 & 12 & 20 & r^4 \\
-1 & 2 & 3 & 4r^2 & 5r^3
\end{bmatrix}
\]

and

\[
\{c\}^T = [c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8].
\]

(15)

Substituting eqn (14) into (10),

\[
U = \frac{1}{2} \pi h \{c\}^T [F]^T [E] [F] \{c\} \, dr.
\]  

(16)
Bending of bimodulus annular plates

Table 1. Convergence study

<table>
<thead>
<tr>
<th>No. of elements</th>
<th>Clamped</th>
<th></th>
<th>Simply supported</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>At C</td>
<td>At A</td>
<td>At B</td>
</tr>
<tr>
<td></td>
<td>$\alpha$</td>
<td>$\beta$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>4</td>
<td>0.1753</td>
<td>-0.1194</td>
<td>0.5771</td>
</tr>
<tr>
<td>6</td>
<td>0.1735</td>
<td>-0.1200</td>
<td>0.5636</td>
</tr>
<tr>
<td>8</td>
<td>0.1726</td>
<td>-0.1215</td>
<td>0.5762</td>
</tr>
<tr>
<td>10</td>
<td>0.1731</td>
<td>-0.1223</td>
<td>0.5625</td>
</tr>
<tr>
<td>12</td>
<td>0.1720</td>
<td>-0.1226</td>
<td>0.5776</td>
</tr>
</tbody>
</table>

$E_1 = 0.5 \times 10^6$ psi, $E_2 = 1.0 \times 10^6$ psi, $\nu_1 = 0.1$, $\nu_2 = 0.2$, $a = 50$ in., $b = 5$ in., $h = 1$ in. (see Fig. 2).

It can be shown that

$$\{\delta\} = [T] \{c\}$$

where

$$\{\delta\}^T = [(u, w, w, w)], (u, w, w, w, w)]$$

$$\{c\} = [T]^{-1}, \quad \{\delta\} = [T1N][\delta].$$

Substituting eqn (17) into (15)

$$U = \frac{1}{2}nh[\delta]^T[TIN]^T[\{F\}]^T[E][\{F\}]r$$

$$\times dr[TIN][\delta].$$

Hence the stiffness matrix of the annular element is

$$[K] = 2nh[TIN]^T[\{F\}]^T[E][\{F\}]r$$

$$\times dr[TIN].$$

EXTERNAL WORK DONE

The external work done by lateral uniformly distributed load "p" is:

$$W = h \int \{W\}_r \text{pr} \, dr \, d\theta$$

where

$$\{W\} = [F]v_1 \{c\}_w$$

$$\{c\}_w = [TIN]_w \{\delta\}_w$$

$$\{c\}_w^T = [c_1 c_2 c_3 c_4 c_5 c_6]$$

$$\{\delta\}_w^T = [(w, w, w), (w, w, w, w)]$$

$$W = 2nhp[\delta]_w^T[TIN]^T[\{F\}]_r dr.$$ 

Load vector

$$\{P\} = -2nhp[TIN]^T[\{F\}]_r dr.$$ 

Thus the equation of equilibrium for an element can be written as:

$$[K][\delta] = [P].$$

After assembling the stiffness matrices and load vectors, the geometrical boundary conditions are applied. Solving the above equilibrium equations, the nodal displacements and their derivatives can be found.

The force-resultants and moment-resultants are given by:

$$\{N\} = [A] \{B\} \{\delta\}$$

$$\{M\} = [B]^T \{D\} \{\delta\}.$$ 

NUMERICAL WORK AND DISCUSSION

For numerical work, annular circular plate with the following material properties, $E_1 = 0.5 \times 10^6$ psi, $E_2 = 0.1 \times 10^6$ psi, $\nu_1 = 0.10$, $\nu_2 = 0.20$, is considered.

Convergence study is carried out with different numbers of elements and the results together with the details of the plates are presented in Table 1. From this table, it can be noted that there is good convergence of solution. For further investigations mesh with ten elements is considered.

In order to check the validity of the programme isotropic annular plates have been analysed and the results are compared with the classical plate theory solution given by Timoshenko. The results are given in Table 2 for: (i) both the edges clamped and (ii) both the edges simply supported boundary conditions. They are found to agree well.

Parametric study for the bimodulus plates with various $b/a$ ratios is carried out for clamped and simply supported boundary conditions. The variations of inplane and lateral displacements, the forces and moments, are presented for clamped plates in Figs 4–6 and for simply supported plates in Figs 7–9. Figure 10 shows the positions of the neutral surface for the clamped and simply supported plates. To the
Table 2

<table>
<thead>
<tr>
<th>r (in.)</th>
<th>α, (This work)</th>
<th>β, (This work)</th>
<th>α, (Timoshenko)</th>
<th>β, (Timoshenko)</th>
<th>α, (This work)</th>
<th>β, (This work)</th>
<th>α, (Timoshenko)</th>
<th>β, (Timoshenko)</th>
</tr>
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<tr>
<td>5.0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.7605</td>
<td>0.7662</td>
<td>0.0000</td>
<td>0.0000</td>
<td>-0.4725</td>
<td>-0.4788</td>
</tr>
<tr>
<td>9.5</td>
<td>0.0473</td>
<td>0.0473</td>
<td>0.1242</td>
<td>0.1255</td>
<td>0.2918</td>
<td>0.2918</td>
<td>-0.2667</td>
<td>-0.2681</td>
</tr>
<tr>
<td>14.0</td>
<td>0.1262</td>
<td>0.1262</td>
<td>-0.0697</td>
<td>-0.0697</td>
<td>0.5264</td>
<td>0.5264</td>
<td>-0.3187</td>
<td>-0.3187</td>
</tr>
<tr>
<td>18.5</td>
<td>0.1927</td>
<td>0.1927</td>
<td>-0.1514</td>
<td>-0.1514</td>
<td>0.6966</td>
<td>0.6965</td>
<td>-0.3644</td>
<td>-0.3644</td>
</tr>
<tr>
<td>23.0</td>
<td>0.2295</td>
<td>0.2295</td>
<td>-0.1764</td>
<td>-0.1765</td>
<td>0.7935</td>
<td>0.7935</td>
<td>-0.3831</td>
<td>-0.3831</td>
</tr>
<tr>
<td>27.5</td>
<td>0.2312</td>
<td>0.2311</td>
<td>-0.1634</td>
<td>-0.1634</td>
<td>0.8133</td>
<td>0.8133</td>
<td>-0.3737</td>
<td>-0.3737</td>
</tr>
<tr>
<td>32.0</td>
<td>0.2003</td>
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<td>-0.1202</td>
<td>0.7579</td>
<td>0.7578</td>
<td>-0.3380</td>
<td>-0.3380</td>
</tr>
<tr>
<td>36.5</td>
<td>0.1455</td>
<td>0.1454</td>
<td>-0.0514</td>
<td>-0.0514</td>
<td>0.6344</td>
<td>0.6344</td>
<td>-0.2779</td>
<td>-0.2779</td>
</tr>
<tr>
<td>41.0</td>
<td>0.0807</td>
<td>0.0807</td>
<td>0.0408</td>
<td>0.0408</td>
<td>0.4550</td>
<td>0.4550</td>
<td>-0.1951</td>
<td>-0.1951</td>
</tr>
<tr>
<td>45.5</td>
<td>0.0245</td>
<td>0.0245</td>
<td>0.1545</td>
<td>0.1546</td>
<td>0.2365</td>
<td>0.2365</td>
<td>-0.0906</td>
<td>-0.0905</td>
</tr>
<tr>
<td>50.0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.2888</td>
<td>0.2888</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0348</td>
<td>0.0348</td>
</tr>
</tbody>
</table>

\( E = 0.5 \times 10^9 \text{ psi}, \ v = 0.33, \ a = 30, \ b = 5, \ h = 1 \) (isotropic material).
\( a = w \times 10^{\text{in.}}, \ \beta = \frac{d^2w}{dr^2} \times 10^9/\text{in.} \)

Fig. 4. Variation of displacements—clamped annular plate.

Fig. 5. Variation of moments—clamped annular plate.
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Fig. 6. Variation of forces—clamped annular plate.

Fig. 7. Variation of displacements—simply supported annular plate.

Fig. 8. Variation of moments—simply supported annular plate.
authors' knowledge, there is no other result available in the literature to compare with the present one.

In this study a bimodulus annular circular plate, subjected to lateral load, has been analysed using the finite element method. The results for a homogeneous-material plate agree well with the existing results. The method can be applied to other types of loading and boundary conditions and is being extended to annular laminated bimodulus plates.

REFERENCES