ONE-DIMENSIONAL WAVE PROPAGATION IN MATERIALS WITH DIFFERENT MODULI IN TENSION AND COMPRESSION

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Abstract—One-dimensional wave propagation in compressible and incompressible elastic materials behaving differently in tension and compression is investigated. The constitutive equations of these materials are nonlinear even though small deformations are considered. The characteristic wavespeeds are derived, the hyperbolicity condition is investigated, and analytical simple wave solutions are obtained in a compressible and incompressible semi-infinite half-space. The presented solutions exhibit interesting phenomena of wave propagation, like that of a coupled shear-normal plane wave propagating steadily with a constant velocity in a compressible medium with different moduli in tension and compression.

INTRODUCTION

A general theory for elastic solids with different moduli in tension and compression has been recently developed by Green and Mkrtichian in[1]. The theory has been systematically developed in that paper for finite elastic deformations and then specialized for the case of infinitesimal strains. In[1], the reader can find a selected list of references which deal with the theoretical and practical aspects of such materials. Another recent work in the field is that by Jones[2], where additional information can be found on materials with different moduli in tension and compression used in practice.

A very characteristic feature of the theory presented in[1] is that even for the case of infinitesimal deformations the constitutive equations are nonlinear. Furthermore, their form depends on the signs of the principal strains, and at every point of the medium one of four possible forms of the constitutive relations applies. The complexity of the constitutive theory of these materials makes it necessary that the solution of most of the boundary value problems be carried out by numerical or iterative schemes. For example, an iterative technique has been used in[3], where the Boussinesq problem for a material with different behaviour in tension and compression has been solved.

To the best knowledge of the author, the investigation of wave propagation phenomena in the framework of the constitutive theory presented by Green and Mkrtichian in[1] has not yet been carried out. In this paper we treat 1-dimensional plane waves in compressible and incompressible elastic media as modeled in[1], and in the context of infinitesimal strains. The considered displacement field is in the form $u_i = u_i(x_1,t)$ where $u_i$, with $i = 1,2,3$, denote the three displacement components and they depend on one space coordinate $x_1$, and the time $t$. Analytical solutions are obtained, which, it is hoped will serve a twofold purpose: first by their explicitness, they will help for a better understanding of the mechanical behaviour of these materials in dynamic situations; second, they will serve as a check for numerical and iterative methods which will be unavoidable in more complicated problems.

The paper contains three sections. In the first section, a very brief summary of the constitutive theory as presented in[1] is given. Only the part of the theory concerned with small deformations is reviewed. In the second section, 1-dimensional plane waves in a compressible medium with different moduli in tension and compression are treated. The characteristic wavespeeds are obtained and the hyperbolicity condition is investigated in detail. Since the constitutive relations are nonlinear as previously mentioned, the characteristic wavespeeds are functions of the displacement gradients. This phenomenon is in a way similar to that encountered in wave propagation in classical elastic solids at large deformations. However, the situation here is rather special in the sense that, for the considered 1-dimensional motions, this dependence on the displacement gradients $m_i = \partial u_i/\partial x_1$, $u_i$, with $i = 1,2,3$, is only through the ratio
\[ r = \sqrt{m_2^2 + m_3^2}/m_1. \] Restrictions are obtained on the material parameters in order that the hyperbolicity condition be fulfilled irrespective of the value of \( r \).

In the second part of the second section, a class of 1-dimensional motions, called simple waves, is investigated for the present material. The theory of simple waves was derived by Bland for nonlinear elastic materials under large deformations\[4\]. We show here that this theory is applicable as well to wave propagation at infinitesimal strains in compressible elastic materials with different moduli in tension and compression. One circularly polarized wave and two plane polarized waves propagating in the positive \( x_1 \) direction are obtained. An analytical description is given of the circularly polarized wave and two particular analytical solutions are given for plane polarized waves. One plane polarized wave is rather unique, as it contains a shear and a normal component propagating steadily with a constant speed of propagation. Such a situation in which a mixed shear-normal steady wave propagates with a constant velocity of its own is not encountered in a classical elastic material. The second section terminates with a description of the initial and boundary conditions in a semi-infinite compressible half-space suited in a form to generate circularly and plane polarized waves.

In the third section, 1-dimensional wave propagation in an incompressible medium with different moduli in tension and compression is considered. It is shown that the shear wave propagation in such an incompressible medium, which is initially undeformed, is linear. This linear shear wave propagation is similar to that in a classical linear elastic material except in the fact that it is accompanied in the present case by stress components other than the shear stress, a phenomenon which does not exist in the corresponding classical case. In the second part of the third section it is shown that such shear propagation can become nonlinear if initial strains in some special directions are present. The equation of motion which governs the transverse displacement in that case is nonlinear and a pulse applied to the surface of a semi-infinite half space may spread out or converge into a shock, depending on specific conditions which are investigated in detail for the present material.

### I. The Constitutive Equations for Materials with Different Moduli in Tension and Compression

We present here a brief summary of the theory proposed by Green and Mkrichian in\[1\]. The basic hypothesis of the theory is that the elastic response of the body depends on the signs of the principal strains. At a given point of the medium, the continuum behaves as a classical elastic body if all the principal strains are either positive or negative at that point. When one principal strain differs in sign from the other two, it is assumed that the strain energy will depend on the unit principal vector \( \mathbf{a} \) which corresponds to that specific principal strain.

At any point of the medium the principal strains \( \mu_1, \mu_2 \) and \( \mu_3 \) are chosen such that \( \mu_1 \geq \mu_2 \geq \mu_3 \). It is shown in\[1,5\] that the constitutive law for a compressible material with different moduli in tension and compression becomes

\[
\begin{align*}
(a) & \quad \mu_1 \geq \mu_2 \geq \mu_3 \geq 0 \\
T &= \alpha(\text{tr} \, \varepsilon) \mathbf{I} + 2\beta \varepsilon,
\end{align*}
\]

\[
\begin{align*}
(b) & \quad \mu_1 \geq \mu_2 \geq 0 \geq \mu_3 \\
T &= \alpha(\text{tr} \, \varepsilon) \mathbf{I} + 2\beta \varepsilon + 2\nu \mu_3 \varepsilon_3 \varepsilon_3,
\end{align*}
\]

\[
\begin{align*}
(c) & \quad 0 \geq \mu_1 \geq \mu_2 \geq \mu_3 \\
T &= -\alpha(\text{tr} \, \varepsilon) \mathbf{I} + 2(\beta + \nu) \varepsilon - 2\nu \mu_1 \varepsilon_1 \varepsilon_1,
\end{align*}
\]

\[
\begin{align*}
(d) & \quad 0 \geq \mu_1 \geq \mu_2 \geq \mu_3 \\
T &= \alpha(\text{tr} \, \varepsilon) \mathbf{I} + 2(\beta + \nu) \varepsilon.
\end{align*}
\]

In the above equations \( T \) is the Cauchy stress tensor, \( \mathbf{I} \) is the unit matrix and \( \varepsilon \) is the infinitesimal strain tensor whose components are given by

\[
\begin{align*}
\varepsilon_{ij} &= \left( \frac{\partial}{\partial x_j} u_i + \frac{\partial}{\partial x_i} u_j \right)/2,
\end{align*}
\]
with \( u_i \) being the displacements and \( x_i \) being the space coordinates. In the same equations \( \alpha, \beta \) and \( \nu \) are the material constants, and \( g_i \) with \( i = 1,2,3 \) are the unit vectors of the principal directions of the strain tensor corresponding to the principal strains \( \mu_i \). It is shown in [1] that \( g_1 g_1 \) is given by

\[
g_1 g_1 = (\varepsilon_2 - (\mu_2 + \mu_3)\varepsilon + \mu_2\mu_3\varepsilon^2)/((\mu_1 - \mu_2)(\mu_1 - \mu_3)),
\]

with similar expressions for \( g_2 g_2 \) and \( g_3 g_3 \) being obtained by cyclic change of \( \mu_1, \mu_2 \) and \( \mu_3 \). Expression (6) is obtained as a result of differentiating a principal strain with respect to the strain components.

We note that for the case of all the principal strains being either positive or negative, i.e. eqns (1) and (4), the material behaves like a classical elastic material with equivalent Lamé parameters \( \alpha \) and \( \beta \), eqn (1), or \( \alpha \) and \( (\beta + \nu) \), eqn (4). The constant \( \nu \) can be taken as that characterizing the different behaviour of the material in tension and compression and for \( \nu = 0 \), eqns (1)-(4) become identical.

When the body is incompressible, \( tr \varepsilon = 0 \), \( \mu_1 + \mu_2 + \mu_3 = 0 \), and it is shown in [1, 5] that the eqns (1)-(4) reduce to the following two cases

(i) \[
\mu_1 \geq \mu_2 \geq \mu_3 \Rightarrow \frac{T}{p} = \frac{1}{2} \beta \varepsilon + 2\nu \mu_3 g_3 g_3,
\]

(ii) \[
\mu_1 \geq 0 \geq \mu_2 \Rightarrow \frac{T}{p} = \frac{1}{2} (\beta + \nu) \varepsilon - 2\nu \mu_1 g_1 g_1,
\]

where \( \beta \) is an arbitrary function of position and time.

It should be noted that even though the considered deformations are infinitesimal, the constitutive eqns (1)-(4) and (7) and (8) are nonlinear in the strains \( \varepsilon_{ij} \) and a linearization process is not possible due to the form of the expressions given in (6).

2. ONE-DIMENSIONAL WAVE PROPAGATION IN A COMPRESSIBLE MEDIUM BEHAVING DIFFERENTLY IN TENSION AND COMPRESSION

2(a) The equations of motion and the characteristic wavespeeds

Let us consider 1-dimensional deformations in the form

\[
u_i = u_i(x_1,t),
\]

with \( i = 1,2,3 \), and \( t \) denotes the time.

The infinitesimal strain tensor is given by

\[
\varepsilon = \begin{bmatrix}
m_1 & m_2/2 & m_3/2 \\
m_2/2 & 0 & 0 \\
m_3/2 & 0 & 0
\end{bmatrix}
\]

where \( m_i = \partial / \partial x_1 u_i \).

The principal strains at any point of the medium are

\[
\mu_1 = (m_1 + \sqrt{m_1^2 + N})/2, \quad \mu_2 = 0,
\]

\[
\mu_3 = (m_1 - \sqrt{m_1^2 + N})/2, \quad \text{where } N = m_2^2 + m_3^2.
\]

We note that in the above equation no matter what the sign of \( m_1 \), if \( N \neq 0 \) then \( \mu_1 > 0 \) and \( \mu_3 \leq 0 \). Therefore, for the 1-dimensional motion to be investigated, the behaviour of the medium simplifies in the sense that only one zone of the constitutive equation applies, eqn (3), but yet it remains to be nonlinear in the displacement gradients \( m_i \).
The equations of motion are given by
\[
\frac{\partial}{\partial x_1} T_{ij} = \rho \frac{\partial^2}{\partial t^2} u_i, \tag{12}
\]
where \(\rho\) is the density of the material.

Using eqn (3) together with (10) and (11) we obtain the following expressions for the stresses \(T_{ij}\)
\[
\begin{align*}
T_{11} &= k_1 m_1 - \nu(2m_1^2 + N + 2m_1\sqrt{m_1^2 + N})/2\sqrt{m_1^2 + N}, \\
T_{12} &= T_{21} = k_2 m_2 - \nu((m_1 m_2 + m_2\sqrt{m_2^2 + N})/2\sqrt{m_2^2 + N}, \\
T_{13} &= T_{31} = k_2 m_3 - \nu((m_1 m_3 + m_3\sqrt{m_3^2 + N})/2\sqrt{m_3^2 + N}, \\
T_{22} &= -k_1 m_1 - \nu(2m_1^2 + N), \\
T_{23} &= T_{32} = -k_2 m_2 m_3/\sqrt{m_1^2 + N}, \\
T_{33} &= \nu(2m_1^2 + N), \\
T_{ij} &= T_{ji}.
\end{align*}
\tag{13}
\]

where \(k_1 = \alpha + 2(\beta + \nu), \ k_2 = (\beta + \nu).\) We note that for \(\nu = 0\), the classical linear elasticity equations are obtained. Furthermore, it is observed that the shear stresses \(T_{22} = T_{33}\) are nonvanishing in this 1-dimensional problem of a material behaving differently in tension and compression. Finally it is seen that although the deformation gradients \(m_i\) are considered to be small, no linearization is possible in eqns (13) and (14).

It is shown in [1] that the constitutive relations (1)–(4) are in fact derived from a set of strain energies through the relation
\[
T_{ij} = (1/2)\left(\frac{\partial}{\partial \varepsilon_{ij}} W + \frac{\partial}{\partial \varepsilon_{ji}} W\right). \tag{15}
\]
The strain energy corresponding to eqn (3) reads
\[
W = (1/2) \alpha (\varepsilon \varepsilon)^2 + (\beta + \nu) \varepsilon^2 - \nu \mu^2, \tag{16}
\]
which reduces for the present 1-dimensional case to
\[
W = (1/2) \alpha m_1^2 + (\beta + \nu)[m_1^2 + (N/2)] - \nu((m_1 + \sqrt{m_1^2 + N})^2/4). \tag{17}
\]
It is a simple matter to verify that the stresses \(T_{11}, T_{21}, T_{31}\) given in eqn (13) can also be written as
\[
T_{ij} = \frac{\partial}{\partial m_i} W, \ j = 1,2,3. \tag{18}
\]
The equations of motion (12) take then the form
\[
\left(\frac{\partial^2}{\partial m_i \partial m_j} W\right) \frac{\partial^2}{\partial x_1^2} u_i = \rho \frac{\partial^2}{\partial t^2} u_i, \ i = 1,2,3. \tag{19}
\]
We define the function \(\Phi\) as
\[
\Phi = W/\rho \ \text{with} \ \Phi_{ij} = -\frac{\partial^2}{\partial m_i \partial m_j} \Phi. \tag{20}
\]
The characteristic wave speeds \(\lambda\) are then given by
\[
\begin{vmatrix}
\Phi_{11} - \lambda^2 & \Phi_{12} & \Phi_{13} \\
\Phi_{21} & \Phi_{22} - \lambda^2 & \Phi_{23} \\
\Phi_{31} & \Phi_{32} & \Phi_{33} - \lambda^2
\end{vmatrix} = 0. \tag{21}
\]
Bland[4] shows that for a function $\Phi$ depending on $m_l$ and $N$ as we presently have, the cubic algebraic equation in $\lambda^2$ factorizes into

$$\lambda^2 - 2\Phi_N[\lambda^4 - \Phi_{m,m} + 4N\Phi_{NN} + 2\Phi_N]\lambda^2 + 2(2N\Phi_{m,m} \Phi_{NN} + \Phi_{m,m} \Phi_N - 2N^2 \Phi_{m,m,m}^2)] = 0,$$

(22)

where $\Phi_N = \partial\Phi/\partial N$. Thus we obtain for the characteristic wavespeeds the following expressions

$$\lambda_1^2 = 2\Phi_N,$$

(23)

$$\lambda_2^2 = (1/2)\Phi_{m,m} + 2N\Phi_{NN} + \Phi_N \pm \sqrt{(- (1/2)\Phi_{m,m} + 2N\Phi_{NN} + \Phi_N^2 + 4\Phi_{m,m}^2)}^{1/2}.$$  

(24)

We make now the definitions $n^2 = N$, and $m = m_1$, and easily prove that

$$\lambda_2^2 = [(\Phi_{mm} + \Phi_{nn})/2] \pm [(1/2)[(\Phi_{mm} - \Phi_{nn})^2 + 4\Phi_{m,m}^2]^{1/2}.$$  

(25)

In order to obtain real wave speeds, that is, the equations to be hyperbolic, we require

$$\lambda_i^2 > 0 \text{ for } i = 1,2,3.$$  

It is seen from (25) that $\lambda_2^2 > 0$ demands

$$(\Phi_{mm} + \Phi_{nn})^2 > [(\Phi_{mm} - \Phi_{nn})^2 + 4\Phi_{m,m}^2],$$

(26)

which gives

$$\Phi_{mm} + \Phi_{nn} > \Phi_{m,m}.$$  

(27)

The hyperbolicity conditions, therefore, reduce to $\lambda_i^2 > 0$ together with eqn (27). In the present case we obtain after some manipulations

$$\Phi_N = \left\{[(\beta/2) + [(\nu/4)(\sqrt{1 + \beta^2} - 1)](\sqrt{1 + r^2})]/\rho \right\},$$

$$\Phi_{mm} = [2\gamma_1 - \left\{\gamma_2(2 + 3\beta^2)/(1 + r^2)^{3/2}\right\}]/\rho,$$

$$\Phi_{nn} = [2\gamma_2 - \left\{\gamma_1(1 + r^2)^{3/2}\right\}]/\rho,$$

(28)

where $r = n/m$ and $\gamma_1 = (\alpha/2) + \beta + (\nu/2)$, $\gamma_2 = (\beta/2) + (\nu/4)$, $\gamma_3 = (\nu/2)$.

We note here the interesting fact that, due to the form of expressions given in eqn (28), the characteristic speeds depend on the ratio $r$ of the displacement gradients. This dependence of the wavespeeds on the displacement gradients is well known in problems of wave propagation at large strains in classical elastic nonlinear materials. The situation here though, is rather unique in the sense that this dependence is through the ratio $r$ of the displacement gradients. In the classical nonlinear case, if the material considered reduces to Hooke's law for small strains, hyperbolicity can always be insured by demanding the displacement gradients to be small enough. In the present case, concerning the hyperbolicity requirements, two possibilities exist. It has been already mentioned that for $\nu = 0$ Hooke's law is obtained. Therefore, as a first possibility we have that for $(\nu/\alpha) \ll 1$, $(\nu/\beta) \ll 1$, no matter what the value of $r$, hyperbolicity is always insured. As a second possibility we consider the case where $\alpha$, $\beta$ and $\nu$ are given, and $r$ is comparable in magnitude to $\alpha$ and $\beta$, and demand that the hyperbolicity conditions be fulfilled irrespective of the loading conditions in a 1-dimensional problem, for instance, that of a semi-infinite half space under uniform loading. In this case bounds on $\alpha$, $\beta$ and $\nu$ need to be found such that the wavespeeds are real for any value of $r$.

From eqn (23) and the first of (28) we see that

$$\lambda_1^2 = [(\beta + (\nu/2)f(r))/\rho].$$

(29)
where

\[ f(r) = (\sqrt{1 + r^2} - 1) / \sqrt{1 + r^2}. \quad (30) \]

We note that \( 0 < f(r) \leq 1 \) and \( f(0) = 0 \). It follows that for \( \lambda_i^2(r = 0) > 0 \) we need \( \beta > 0 \).

Furthermore, for any \( r \) we demand \( \lambda_i^2(r) > 0 \), that is

\[ \nu > - 2\beta / f(r). \quad (31) \]

Since \( 0 < f(r) \leq 1 \), clearly if \( \nu > -2\beta \) then eqn (31) is automatically satisfied.

The investigation of the hyperbolicity condition (27) is more complicated. Using eqn (27) and the last three of (28) we obtain after considerable manipulation the following hyperbolicity condition

\[ D > 0, \quad (32) \]

where \( D = \tilde{\nu}^2 A + \tilde{\nu} B + C \), in which we have introduced the non-dimensional parameter \( \tilde{\nu} = \nu / \beta \) and

\[
A = [(1 + r^2)/2] - [(1 + r^7)/2] - [(2 + 3r^7)(1 + r^7)/4] + [(2 + 3r^7)/4] - [r^8/4],
\]

\[
B = [(\bar{\alpha} \bar{r})^2 + 2][1 + r^2] - [(\bar{\alpha} \bar{r}) + 1][1 + r^2] - [(2 + 3r^2)(1 + r^2)/2] - [(2 + 3r^2)(1 + r^2)/2],
\]

\[
C = [2 + \bar{\alpha}][1 + r^2]^3, \quad \text{where} \quad \bar{\alpha} = (\alpha / \beta).
\]

Let us first investigate the case of \( r = 0 \).

In this case we have: \( A(0) = 0, B(0) = 0, \) and \( C(0) = 2 + \bar{\alpha} \). Therefore, we need

\[ 2 + \bar{\alpha} = [(2\beta + \alpha) / \beta] > 0. \quad (34) \]

Since \( \beta > 0 \), this implies that the following condition needs to be satisfied

\[ (2\beta + \alpha) > 0. \quad (35) \]

For \( r > 0 \) we first note that \( A, B \) and \( C \) are functions of \( r^2 \) and it is sufficient to consider values of \( 0 < r^2 < +\infty \). First, we note that the asymptotic value of \( A \) as \( r^2 \to \infty \) is \( A \sim (r^4/4) \). A numerical investigation of \( A(r^2) \) shows that \( A > 0 \) for any nonzero value of \( r \). Therefore, if there are real roots \( \bar{\nu}_1 > \bar{\nu}_2 \) of

\[ \tilde{\nu}^2 A + \tilde{\nu} B + C = 0 \quad (36) \]

then, for \( D > 0 \), we need \( \bar{\nu} > \bar{\nu}_1 \) and \( \bar{\nu} < \bar{\nu}_2 \), for any given value of \( \bar{\alpha} \) and for any \( r \). And if there exists no real roots, then, since \( A > 0 \), the condition \( D > 0 \) is automatically insured.

We investigate first the asymptotic behaviour of the condition \( D > 0 \) as \( r \to \infty \). It is noted that the asymptotic values of \( A, B \) and \( C \) as \( r \to \infty \) are

\[ A \sim (1 / 4) r^6, \quad B \sim [(\bar{\alpha} / 2) + 2] r^6, \quad C \sim (2 + \bar{\alpha}) r^6. \quad (37) \]

We see therefore that eqn (36) in this case reduces to

\[ r^4(\nu^2 + \nu(8 + 2\bar{\alpha}) + 4(\bar{\alpha} + 2)) = 0. \quad (38) \]

The discriminant of the above quadratic equation is \( \Delta = 4(\bar{\alpha}^2 + 4\bar{\alpha} + 8) \). Since

\[ (\bar{\alpha}^2 + 4\bar{\alpha} + 8) > (\bar{\alpha}^2 + 4\bar{\alpha} + 4) = (\bar{\alpha} + 2)^2 > 0, \quad (39) \]
then there always exists two real roots $\tilde{\nu}_1$ and $\tilde{\nu}_2$ which are given by

$$
\begin{align*}
\tilde{\nu}_1 &= -4 - \alpha + \sqrt{\alpha^2 + 4\alpha + 8}, \\
\tilde{\nu}_2 &= -4 - \alpha - \sqrt{\alpha^2 + 4\alpha + 8}.
\end{align*}
$$

(40)

For $D > 0$ we need therefore, $\tilde{\nu} > \tilde{\nu}_1$ and $\tilde{\nu} < \tilde{\nu}_2$. Since $2 + \alpha > 0$ by eqn (34), we note that

$$
\frac{\partial}{\partial \alpha} \tilde{\nu}_2 = \{(-\sqrt{\alpha^2 + 4\alpha + 8} - (\alpha + 2))(\sqrt{\alpha^2 + 4\alpha + 8} + 8)\} < 0,
$$

(41)

and for $2 + \alpha > 0$, $(\tilde{\nu}_2)_{\text{max}} = -4$. Since we have already seen that $\tilde{\nu} > -2$ by eqn (31), the inequality $\tilde{\nu} < \tilde{\nu}_2$ is not applicable and we need only to investigate $\tilde{\nu} > \tilde{\nu}_1$. We note again that

$$
\frac{\partial}{\partial \alpha} \tilde{\nu}_1 = \{(-\sqrt{\alpha^2 + 4\alpha + 8} + (\alpha + 2))(\sqrt{\alpha^2 + 4\alpha + 8} + 8)\} < 0,
$$

(42)

and for $2 + \alpha > 0$, $(\tilde{\nu}_1)_{\text{max}} = 0$, and the restriction $\tilde{\nu} > 0$ would be needed. If we restrict ourselves, however, to positive values of $\alpha > 0$, then the condition on $\tilde{\nu}$ becomes $\tilde{\nu} > \tilde{\nu}_1(0) = -1.172$.

For a general investigation of the hyperbolicity condition $D > 0$, once the parameters $\alpha$ and $\tilde{\nu}$ given, computations need to be made for different values of $r$ and insure that the parameter $\tilde{\nu}$ lies outside the roots $\tilde{\nu}_1$ and $\tilde{\nu}_2$. For example we have chosen $\alpha = 1$ and have obtained the roots $\tilde{\nu}_1$ and $\tilde{\nu}_2$ for values of $0 < r < 10^3$. At that upper limit eqn (36) already reaches its asymptotic form which has been investigated separately. Results show that for $\alpha = 1$ the most restrictive conditions on $\tilde{\nu}$ are $\tilde{\nu} > -1.3948$ and $\tilde{\nu} < -8.6292$. By making more runs for values of $0 < \alpha < 10$, we have come to the conclusion that for positive values of $\alpha$ which are in this range at least, a restriction on $\tilde{\nu}$ in the form of $\tilde{\nu} > 1$ is sufficient for the satisfaction of the hyperbolicity condition. In the sequel we will limit ourselves to positive values of $\alpha$ and, therefore, to $\tilde{\nu} > -1$.

2(b) Simple waves in a compressible medium behaving differently in tension and compression

In this section we start by giving briefly the results obtained by Bland in[4] for some specific classes of 1-dimensional wave propagation in a nonlinearity elastic material under large deformations. It should be again noted that we are dealing in this paper with small deformations; yet it is seen that the equations of motion (19) are nonlinear in $m_i$ and stresses $T_{ij}$ were derivable by eqn (18) from a strain energy $W(m_i, N)$ where $N$ was defined as $N = m^2 + m^3$. This framework is exactly the same as that which exists in the work of Bland and his results are readily applicable to the presently treated material. For plane wave propagation in the direction of the positive $x_1$ axis, it has been proved in[4] that there exists three classes of solutions called simple waves in which all the deformation gradients are functions of one another. One wave is called a circularly polarized wave in which $m_1 = C_1$ and $m_2^2 + m_3^2 = N = C_2$, where $C_1$ and $C_2$ are constants. The displacement gradient vector $m_1$ for such a wave has a constant magnitude $(m_1^2 + N)^{1/2}$ and a constant component in the direction of propagation. Its component in a plane parallel to the wave front is of constant magnitude $N^{1/2}$ but of arbitrary direction. The speed of propagation of the wave is given by $\lambda_i^2 = 2\Phi_N$.

The other two cases are called plane polarized waves in which the component of the displacement gradient vector in a plane parallel to the wave front, that is $N^{1/2}$, always has the same direction. The investigation of these plane polarized waves can therefore be carried out in terms of $n = N^{1/2}$ and $m - m_1$. In these plane polarized simple waves the relation between $m$ and $n$ is given by

$$
\frac{d}{dn} m = [(\lambda_i^2 - \Phi_{nn})/\Phi_{mn}] \quad i = 2,3,
$$

(43)

where $\lambda_i^2$ is given in eqn (25). There are, therefore, two plane polarized waves propagating in the positive $x_i$ direction and corresponding to $\lambda_2^2$ and $\lambda_3^2$, respectively.
Let us first investigate the circularly polarized wave in the presently treated material. This wave can be generated in a semi-infinite half-space $x_1 > 0$ by applying proper boundary conditions at its surface. Let the initial conditions be

$$\left[ \frac{\partial}{\partial t} u_i(x_1, t) \right]_{t=0} = T_{i0}, \quad m_i(x_1,0) = m_{i0} \quad i = 1, 2, 3. \quad (44)$$

If stresses $T_{ii}$ as given by eqs (13) are applied at $x_1 = 0$ such that

$$m_i(0, t) = m_{i0}, \quad [m_3(0, t)]^2 + [m_3(0, t)]^2 = m_{30}^2 + m_{30}^2 \quad (45)$$

then a circularly polarized wave propagates in the $x_1$ direction. The speed of propagation is constant and is given for the present material by eqns (29) and (30) with

$$r = (m_{30}^2 + m_{30}^2)^{1/2} / m_{10}. \quad (46)$$

The plane polarized waves are more difficult to investigate due to the nonlinear differential eqn (43). For a material behaving differently in tension and compression however, we are able to give here a particular analytic solution of this class of simple waves. A solution is sought in which we demand $n/m - r^* = \text{constant}$. This implies through eqn (43) that

$$r^* = \Phi_{nn}/(\lambda^2 - \Phi_{nn}) \quad i = 2, 3 \quad (47)$$

where $\lambda^2$, $\Phi_{nn}$ and $\Phi_{mn}$ are now functions of $r^*$. The question then becomes to find a real value of $r^*$ together with a real propagation speed. Substituting $\lambda^2$ from (25) into (47) gives

$$\pm(1/2)r^*[(\Phi_{nn} - \Phi_{nn})^2 + 4\Phi_{nn}^2]^{1/2} = \Phi_{nn} + [(\Phi_{nn} - \Phi_{nn})r^*/2]. \quad (48)$$

Taking the square, substituting the functions $\Phi_{nn}$, $\Phi_{nn}$, $\Phi_{nn}$ from (28) into (48), we obtain after considerable manipulation the roots for $r^*$ together with two corresponding propagation speeds. In the first case

$$r^* = 0 \quad \text{and} \quad \lambda = [(\alpha + 2)/(\beta/\rho)]^{1/2} \quad (49)$$

and in the second case

$$r^* = \pm \sqrt{k^2 - 1} \quad \text{and} \quad \lambda = [(\alpha + 1 + \beta)/(\alpha + 1 + \beta/2)]^{1/2}[(\alpha + 1 + \beta/2)]^{1/2} (\beta/\rho)^{1/2} \quad (50)$$

where $k = 2[(\alpha + 1 + \beta/2)/(\alpha + 1 + \beta/2)]^{1/2}$, and the same value of $\lambda$ in (50) corresponds to both the positive and negative values of $r^*$. We note first that with $\alpha > 0$ and $\beta > 0$, the wave speed as given in (49) is real. Secondly, for $\alpha > 0$ and $\beta > -1$, it is easily checked that $k^2 > 1$ and the value of $\lambda$ given in (50) is real.

The first wave, as given by eqn (49), is a tension wave propagating steadily with a constant velocity. We note here that since eqn (3) has been used, $\mu_1 = 0$ by definition, and the obtained velocity in (49) is that corresponding to a tension wave. In fact, this can be seen by more direct considerations; that is, if $\mu_1 = 0$, $\mu_2 = \mu_3 = 0$ the constitutive eqn (1) which is linear predicts that such a wave is a possible solution in a material behaving differently in tension and compression. Furthermore, if $\mu_1 = 0$, $\mu_2 = \mu_3 = 0$, eqn (4) applies and a steady compression wave is obtained with a propagation velocity $\lambda = [(\alpha + 2 + 2\beta)/(\beta/\rho)]^{1/2}$.

A pure plane polarized shear wave on the other hand is not possible in the present material and is always accompanied by a tension or compression disturbance. The plane polarized wave, as described by eqn (50) is in fact a rather peculiar type of such a combined wave propagating steadily with a constant velocity. It should be noted that there’s no counterpart of this kind of a wave in a classical material.

The obtained two plane polarized waves can also be derived directly in the present case without the formalism of Bland[4], by considering the equations of motion in the $n$ and $m$
components. We consider a plane strain motion in which the only existing displacement gradients are $m_2$ and $m_1$, and we redefine $m = m_1$ and $n = m_2$. For the plane strain case, the equations of motion (19) with $i = 1, 2$ then read as follows

\[
\begin{align*}
\Phi_{mm} \frac{\partial^2}{\partial x_1^2} u_1 + \Phi_{mn} \frac{\partial^2}{\partial x_1 \partial x_2} u_2 &= \frac{\partial^2}{\partial t^2} u_1 \\
\Phi_{mn} \frac{\partial^2}{\partial x_1^2} u_1 + \Phi_{nn} \frac{\partial^2}{\partial x_1 \partial x_2} u_2 &= \frac{\partial^2}{\partial t^2} u_2,
\end{align*}
\]

(51)

It is noted again that $\Phi_{mm}$, $\Phi_{mn}$ and $\Phi_{nn}$ are functions of $r$ only and try for solution, steady plane waves in the form

\[
\begin{align*}
u_1 &= A_1 \Psi(x_1 - ct), \quad u_2 = A_2 \Psi(x_1 - ct)
\end{align*}
\]

(52)

where $A_1$ and $A_2$ are constants and $c$ is a constant speed of propagation. In spite of the nonlinearity in eqn (51), due to the special dependence on $r$, such solutions are admissible and they give exactly the same results as described by eqns (49) and (50). It is, however, interesting to note that these steady wave solutions for the considered material are in fact special cases of the more general simple wave theory as derived by Bland.

We now give initial and boundary conditions in a semi-infinite half-space suited in a form to generate plane polarized waves. The first wave can be generated under the following conditions

\[
\begin{align*}
\left[ \frac{\partial}{\partial t} u_i(x_1, t) \right]_{t=0} &= 0, \quad i = 1, 2; n(x_1, 0) = 0 \\
m(x_1, 0) &= m_0, \quad T_{11}(0, t) = \beta(\alpha + 2) m(0, t), \\
T_{12}(0, t) &= 0, \quad \text{with} \quad m(0, t) = g(t), g(0) = m_0.
\end{align*}
\]

(53)

The initial conditions corresponding to the second plane polarized wave are

\[
\begin{align*}
\left[ \frac{\partial}{\partial t} u_i(x_1, t) \right]_{t=0} &= 0, \quad i = 1, 2; n(x_1, 0) = n_0 \\
m(0, t) &= n_0(\pm \sqrt{k^2 - 1}).
\end{align*}
\]

(54)

As to the boundary conditions they are obtained from eqns (13), in which we designate now $m_1$ by $m$, $m_2$ by $n$, take $m_3 = 0$, and the following relations are maintained at $x_1 = 0$

\[
n(0, t) = j(t), \quad m(0, t) = j(t)(\pm \sqrt{k^2 - 1})
\]

(55)

and $j(0) = n_0$.

It should be noted that there are in fact two kinds of this second plane polarized wave, namely, one with a compression component in the direction of propagation and the other one with a tension component. It is interesting to observe that both kinds propagate with the same velocity. Furthermore it is seen that for a classical elastic material, $\nu = 0$, and it results from (50) that $k \rightarrow \infty$, $m \rightarrow 0$, $\lambda \rightarrow (\beta\nu)^{1/2}$. In other works the second plane polarized wave reduces in this case to a pure shear wave.

This section is concluded by noting that the solutions obtained for plane polarized waves are particular solutions of eqn (43) in the sense that they are generated only under some specific initial and boundary conditions. It is again emphasized that the second plane polarized wave has a shear and normal component propagating with the same velocity, a feature which is not encountered in classical materials.
In this section wave propagation in an incompressible half-space as described by the constitutive eqns (7) and (8) is considered.

3(a) The case of an incompressible half-space undeformed in its initial configuration

For the case of the 1-dimensional motion as described in eqn (9), through the incompressibility condition tr $\varepsilon = 0$, it is first obtained that $u_{1,1} = 0$, and the components of the strain tensor are given by

$$
\varepsilon_{12} = \varepsilon_{21} = m_2/2, \quad \varepsilon_{13} = \varepsilon_{31} = m_3/2,
$$

while all the other components vanish identically. The principal values of $\varepsilon$ in the above equation are given by: $\mu_1 = N^{1/2} > 0$, $\mu_2 = 0$, $\mu_3 = -\mu_1 < 0$ where $N$ is again taken as $N = m_2^2 + m_3^2$. Using either one of the constitutive eqns (7) or (8) the following expressions for the stresses are obtained

$$
T_{12} = T_{21} = (2\beta + \nu)m_2/2, \quad T_{13} = T_{31} = (2\beta + \nu)m_3/2,
$$

$$
T_{23} = T_{32} = (-\nu/2)(m_2m_3)/N^{1/2}, \quad T_{11} = p - [(\nu/2)N^{1/2}],
$$

$$
T_{22} = p - [(\nu/2)m_2^2/N^{1/2}], \quad T_{33} = p - [(\nu/2)m_3^2/N^{1/2}].
$$

The function $p$ can be obtained by substituting $T_{11}$ from (57) into the first of (12), and integrating with respect to $x_i$, which provides

$$
p = (\nu/2)N^{1/2} + C(t),
$$

where $C(t)$ is a function of time which can be determined from the boundary conditions of the specifically considered problem. Let us treat here a half-space $x_1 \geq 0$ initially at rest, undeformed and subjected to surface tractions in the form

$$
T_{ir}(0,t) = 0, \quad T_{i1} = f_i(t) \quad \text{with} \quad i = 2,3
$$

through the first of above equations it follows that $C(t) = 0$, and the stresses $T_{11}$, $T_{22}$ and $T_{33}$, therefore, become

$$
T_{11} = 0, \quad T_{22} = (\nu/2)[N^{1/2} - (m_2^2/N^{1/2})], \quad T_{33} = (\nu/2)[N^{1/2} - (m_3^2/N^{1/2})].
$$

The other two equations of motion together with the stresses $T_{23}$ and $T_{31}$ give

$$
c_0^2 \frac{\partial^2}{\partial x_i^2} u_i = \frac{\partial^2}{\partial t^2} u_i
$$

with $i = 2,3$ and $c_0^2 = (2\beta + \nu)/2p$.

The problem is, therefore, linear in $u_i$ and the solution for the shear stresses $T_{1i}$ is

$$
T_{1i} = f_i(\phi)H(\phi). \quad i = 2,3
$$

with $\phi = t - (x_i/c_0)$ and $H(\phi)$ is the Heaviside step function. When compared to wave propagation in an incompressible classical linear elastic material, the only interesting feature of the present problem remains in the fact that "reaction" stresses $T_{22}$, $T_{33}$ and $T_{23}$ are obtained here, which do not exist in the classical case.

The presented solution in this section is rather straightforward. However, wave propagation in an incompressible medium with different moduli in tension and compression can become...
more complicated and nonlinear in nature if initial deformations of the medium in some special
directions are considered. An example of such a situation will be given in the next section.

3(b) The case of an incompressible half-space deformed in its initial configuration

It is first noted that for the 1-dimensional motions considered in this paper if initial
deformations in the $\varepsilon_{11}$, $\varepsilon_{12}$ and $\varepsilon_{13}$ strains are admitted, both in the compressible and
incompressible cases, the presented solutions remain unchanged. In fact, such initial defor-
mations were present in the circularly polarized and plane polarized waves of the compressible
medium investigated in Section 2(b). It is guessed, however, that a more general type of initial
defformation involving other strains will effect the principal values of the strains and the
character of the problem might change completely.

As an example, we consider in this section a special kind of initial strains which will render
the problem in the incompressible case nonlinear. Let us consider an incompressible half-space
initially at rest but with a homogeneous initial strain state given by

$$\varepsilon_{13}(x_1,0) = -\varepsilon_{11}(x_1,0) = \varepsilon_0 > 0$$

(63)

while the other strains are initially zero. The initial stress state corresponding to (63) can easily
be obtained by using either one of eqns (7) or (8) (we note that $\mu_1 = \varepsilon_0 > 0$, $\mu_2 = 0$ and
$\mu_3 = -\varepsilon_0 < 0$). After using the free surface condition to eliminate $\rho$, the following initial stress is
obtained

$$T_{33}(x_1,0) = (2\nu + 4\beta)\varepsilon_0,$$

(64)

with all the other stresses vanishing identically. For $t \geq 0$ let the surface $x_1 = 0$ be subjected to
the following boundary conditions

$$T_{11}(0,t) = 0, \quad T_{12}(0,t) = h(t), \quad T_{13}(0,t) = 0.$$  

(65)

The loading being uniform and the initial state being homogeneous, all the possible displace-
ments $u_i$ will be a function of $x_1$ only. The incompressibility condition $\nabla \cdot \sigma = 0$ implies on the
other hand that $u_{1,1} - m_1 = 0$. Furthermore, a superimposed displacement $u_3$ is ruled out from
symmetry considerations. In fact, the medium is initially deformed in the normal strains $\varepsilon_{11}$ and
$\varepsilon_{33}$, the applied disturbance is in the direction of $x_3$; thus, there is no preferred sense in the $x_1$
coordinate and, therefore, there is no $u_3$ superimposed displacement. The components of the
strain tensor are, therefore: $\varepsilon_{11} = -\varepsilon_0$, $\varepsilon_{12} = \varepsilon_{21} = m_2/2$, $\varepsilon_{33} = \varepsilon_0$, while the other components
vanish identically.
The principal strains are given by

$$\mu_1 = \varepsilon_0 \geq 0, \quad \mu_2 = (-\varepsilon_0 + \sqrt{\varepsilon_0^2 + m_2^2})/2 \geq 0, \quad \mu_3 = (-\varepsilon_0 - \sqrt{\varepsilon_0^2 + m_2^2})/2 \leq 0.$$  

(66)

The constitutive eqn (7) applies, and since it is symmetric in $\mu_1$ and $\mu_2$, no distinction is made as
to $\mu_1 > \mu_2$ or $\mu_2 > \mu_1$. Using the strain tensor with its principal values given in (66) in the
constitutive eqn (7) the expressions for the stresses are obtained. It turns out that the stresses
$T_{13}$ and $T_{23}$ vanish; the stresses $T_{11}$, $T_{12}$ and $T_{33}$ are functions of $m_2$, $\varepsilon_0$, and contain the
function $p$ which is determined from the condition $\partial / \partial x_1 T_{11} = 0$ and $T_{11}(0,t) = 0$. For the sake
of brevity we give here only the stresses $T_{12}$ which will be now a nonlinear equation in $m_2$, and
from which will, in fact, $m_2$ be determined

$$T_{12} = \beta m_2 + (\nu/2)[m_2 + (m_2 \varepsilon_0/\sqrt{\varepsilon_0^2 + m_2^2})].$$

(67)

It is noted that for $\varepsilon_0 = 0$ this stress reduces to that given in eqn (57). Substitution of (67) into
the second equation of motion (12) gives a nonlinear differential equation in $u_2$ as follows

$$\left[e^*(m_2; \varepsilon_0)\right]^2 \frac{\partial^2}{\partial x_1^2} u_2 = \frac{\partial^2}{\partial t^2} u_2.$$  

(68)
where
\[ c^* = (\beta/\rho) + (\nu/2)[1 + (\epsilon_{0}/(\epsilon_{0}^2 + m_{2}^2)^{1/2})]. \] (69)

Equation (68) is to be solved under the initial and boundary conditions
\[ \left[ \frac{\partial}{\partial t} w(x,t) \right]_{t=0} = 0, \quad m_{2}(x,0), \quad T_{13}(0,t) = h(t). \] (70)

Once \( h(t) \) is given, \( m_{2}(0,t) \) can be solved from (67) in the form of \( m_{2}(0,t) = q(t) \). Before proceeding to the solution it is necessary to make some comments concerning eqn (67). It should be remembered that this expression was derived for the case \( \epsilon > 0 \). A similar derivation can be carried out for \( \epsilon < 0 \). In this case it is only necessary to change the order of the principal strains which will be written this time as
\[ \epsilon_{1} = -\epsilon_{0} + \sqrt{\epsilon_{0}^2 + m_{2}^2}; \quad \epsilon_{2} = \epsilon_{0} \leq 0; \quad \epsilon_{3} = -\epsilon_{0} - \sqrt{\epsilon_{0}^2 + m_{2}^2} \leq 0 \] (71)
and the applicable constitutive equation will be that given in eqn (8). After carrying out the necessary manipulations in turns out that the same eqn (67) is again obtained also for \( \epsilon_{0} = 0 \). The stresses \( T_{13} \) and \( T_{23} \) vanish again but the stresses \( T_{11}, T_{22} \) and \( T_{33} \) will be different this time.

It is first readily checked that for \( \nu > -\beta \) no matter what the sign of \( \epsilon_{0} \),
\[ c^* \rho = \frac{\partial}{\partial m_{2}} T_{12} > 0 \] (72)
is valid, so that hyperbolicity, together with the invertibility of eqn (67) is insured. Furthermore, it can be verified that
\[ \frac{\partial^2}{\partial m_{2}^2} T_{12} - 2c^* \frac{\partial c^*}{\partial m_{2}} = (-3\nu/2)(\epsilon_{0}^2 m_{2}/(\epsilon_{0}^2 + m_{2}^2)^{3/2}), \] (73)
and therefore for \( \nu > 0 \)
\[ \frac{\partial^2}{\partial m_{2}^2} T_{12} \leq 0 \quad \text{for} \quad (m_{2} > 0, \epsilon_{0} \geq 0), \quad \text{or} \quad (m_{2} \leq 0, \epsilon_{0} \leq 0), \] (74)
and
\[ \frac{\partial^2}{\partial m_{2}^2} T_{12} \geq 0 \quad \text{for} \quad (m_{2} \leq 0, \epsilon_{0} \geq 0) \quad \text{or} \quad (m_{2} > 0, \epsilon_{0} \leq 0). \] (75)

Therefore, for \( \epsilon_{0} > 0 \) the material behaves as one of softening type and for \( \epsilon_{0} < 0 \) it is of hardening type. For \( \nu < 0 \) opposite behaviour is obtained.

Turning now to the solution of eqn (68), it is well known that the solution of this equation together with the initial conditions given in (70) and the boundary condition \( m_{2}(0,t) = q(t) \), can be written as a simple wave with spreading or converging characteristics, this depending on the nature of \( q(t) \) and the type of function \( c^* \) given in (69), see Bland[4] for example. It results in the present case that for \( \epsilon_{0} > 0 \), if \( |m_{2}(0,t)| = |q(t)| \) is a monotonically increasing function for \( t \geq 0 \), the solution is a simple wave with spreading characteristics which for \( \epsilon_{0} < 0 \) a monotonically increasing function \( |q(t)| \) leads to a solution with converging characteristics building up eventually into a shock. If an initial deformation \( m_{2}(x,0) = m_{20} \) is admitted, then a monotonically decreasing \( |q(t)| \) for \( t \geq 0 \) will lead to a spreading wave for \( \epsilon_{0} < 0 \), and to shock formation for \( \epsilon_{0} > 0 \). These conclusions are of course valid for \( \nu > 0 \).

Up to the formation of the shock the simple wave solution is written as
\[ m_{2}(x,t) = q(t - |x|/c^*(m_{2}^2;\epsilon_{0})). \] (76)
Before concluding it is interesting to note two limiting forms of the propagation speed given in eqn (69). It is first seen that this equation can be written as

\[ c^* = \left( \frac{2\beta + \nu}{2\rho} \right) + \left( \frac{\nu}{2\rho} \right) \left\{ 1 + \left( \frac{m_2}{\varepsilon_0} \right)^2 \right\}^{1/2}. \]  
(77)

For an initial deformation much smaller than the superimposed shear, that is \((m_2/\varepsilon_0) \to \infty\), eqn (77) implies \(c^* \to (2\beta + \nu)/2\rho\), which is the same speed obtained for the case of wave propagation in an undeformed incompressible solid in Section 3(a). On the other hand, if the superimposed shear is much smaller then the initial deformation, \((m_2/\varepsilon_0) \to 0\), and, therefore, \(c^* \to (\beta + \nu)/\rho\); that is, these small shear disturbances propagate with a constant but different velocity this time.

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