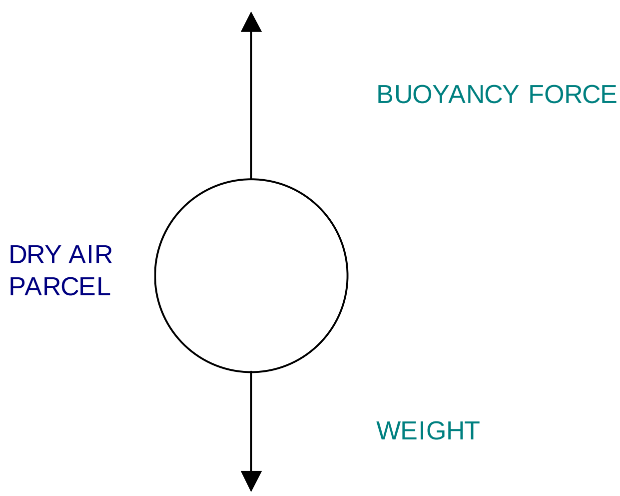
ENVIRONMENTAL PHYSICS COMPUTER LAB #3 :
Stability of Dry Air and Brunt-Vaisala Oscillations

Consider a parcel of dry air of volume V, temperature T and density ρ. It displaces the same volume V of surrounding air of temperature T' and density ρ'. The parcel pushed upwards by the buoyancy force and downwards by its weight. We then wr Newton's second law: $ma = F = F_{buoyancy} - F_{weight}$. Now $m = \rho V$ and $F_{weight} = \rho V g$ According to Archimedes' principle the buoyancy force is the weight of the displace fluid: $F_{buoyancy} = \rho' V g$. By substituting these formulas into Newton's second law we get: $a = g(\rho'/\rho - 1)$. By using the ideal gas equation of state, $\rho = pM/RT$, $\rho' = pM/RT$ we find: $\rho'/\rho = T/T'$. The equality of pressures inside and outside the parcel is required by mechanical equilibrium. Hence the acceleration of the air parcel is: $a = g(T/T'-1)$.



Exercise 1: Consider a parcel of cold air T = 0 °C surrounded by warmer air T = 10 °C. Compute the acceleration of the parcel.

$$a(T, T') := g \cdot \left(\frac{T}{T'} - 1\right)$$

$T := 273$ $\qquad T' := 283$

$a(T, T') = -0.347\, m \cdot s^{-2}$ What is the significance of the minus sign?

Exercise 2: Repeat Exercise 1 with the following changes: parcel of warm air T = 25 °C surrounded by cold air T' = 5 °C.

$T := 273 + 25$ $\qquad T' := 273 + 5$

$a(T, T') = 0.706\, m \cdot s^{-2}$

We now study the motion of the parcel by using Newton's second law and thermodynamic arguments for temperature profiles. The temperature profile for the parcel is T = Tnaught - Γ(z - znaught) where $\Gamma = Mg/c_p$. Γ is called the dry adiabatic lapse rate. This formula is derived by assuming the parcel to be an ideal gas that undergoes an adiabatic process, i.e. no heat exchange between the parcel and the environment. The molar mass for air is 29g/mole and since air is diatomic $c_p = 7R/2$.

$M := 29 \cdot 10^{-3}$ $\qquad g := 9.81$ $\qquad R := 8.31$ $\qquad c_p := \frac{7}{2} \cdot R$

$\Gamma := \frac{M \cdot g}{c_p}$ $\qquad \Gamma = 9.781 \times 10^{-3}$

This is the dry adiabatic lapse rate in K/m. Every 100m the parcel's temperature is lowered by about 1C.

$Tnaught := 273$ $\qquad znaught := 100$

The temperature profile of the parcel: $T(z) := Tnaught - \Gamma \cdot (z - znaught)$

We denote γ the ambient lapse rate. Peixoto and Oort, pg 141 , state that the mean γ is 0.6K/100m, i.e. in average for a 1km climb the temperature drops by 6 C.

$$\gamma := 6 \cdot 10^{-3}$$

The temperature profile of the ambiant air:

$$T'(z) := Tnaught - \gamma \cdot (z - znaught)$$

We solve numerically: a = d^2z/dt^2 = g(T(z)/T'(z)-1) by using the Mathcad solver odesolve(t,Tmax,#), where t is the variable of integration, Tmax is the terminal point of the integration interval, #(optional) is the number of steps used in calculating the solution. Read more about this solver in Help.

At t = 0 the parcel is at z = znaught and has a velocity v = vnaught.

$$vnaught := 1$$

Before writing the equation and the initial conditions, we need to type given.

Given

$$\frac{d^2}{dt^2}z(t) - g \cdot \left[\frac{Tnaught - \Gamma \cdot (z(t) - znaught)}{Tnaught - \gamma \cdot (z(t) - znaught)} - 1\right] = 0$$

$$z(0) = znaught$$

Note you need z prime:

$$z'(0) = vnaught$$

We solve now the differential equation:

$$z := Odesolve(t, 2000)$$

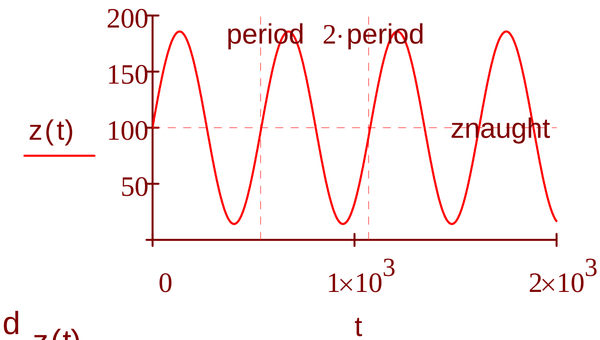
We graph the position z versus time and note that the parcel undergoes so calle
Brunt-Vaisala oscillations. The period of the oscillations is:

$$period := \frac{2\cdot\pi}{\sqrt{g\cdot\frac{\Gamma-\gamma}{Tnaught}}}$$

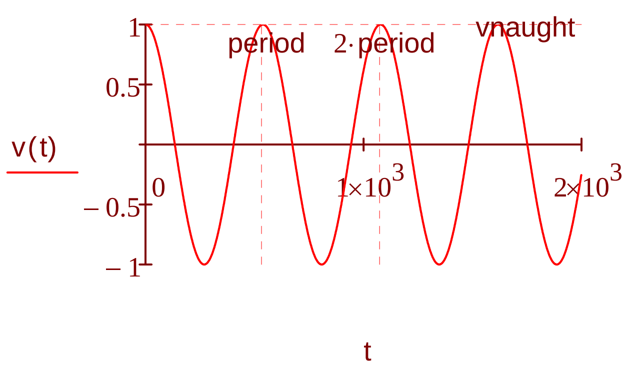
$period = 539.019$ seconds. This is about 9 minutes.

Challenge problem: Prove the Brunt-Vaisala period formula.

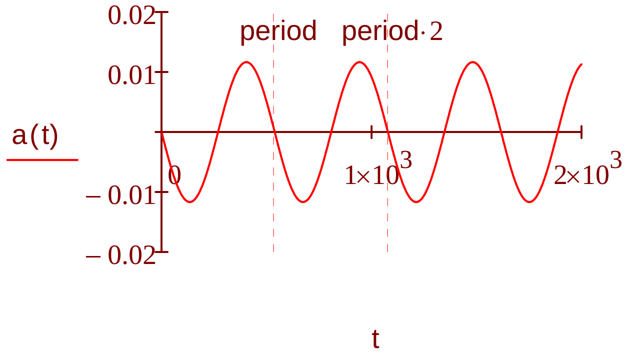
$t := 0 .. 2000$



$$v(t) := \frac{d}{dt}z(t)$$



$$a(t) := g\cdot\left[\frac{Tnaught-\Gamma\cdot(z(t)-znaught)}{Tnaught-\gamma\cdot(z(t)-znaught)}-1\right]$$



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* If $\Gamma > \gamma$, as in our simulation, the parcel undergoes oscillatory motion. The dry air is **stable** against the vertical motion of a parcel. Smaller the difference $\Gamma - \gamma$, longer the period is.
* If $\Gamma < \gamma$ the Brunt-Vaisala period is an imaginay number, meaning that the motion is not oscillatory. The dry atmosphere is **unstable** with respect to vertical motion of a parcel. The parcel does not return to its initial location.

Assignment: Run the simulation for $\gamma = 0.03^{o}C/m$. Do you still observe oscillations?
For an inversion layer the ambient lapse rate γ is negative, as the warmer layer is above the colder layer. Since $\Gamma > 0 > \gamma$ the inversion air is stable against the vertical motion of a parcel. Compute the ambiant lapse rate γ if the period is 3 minutes.

From the Brunt-Vaisala formula we compute γ:

$$\text{period} := \frac{2\cdot\pi}{\sqrt{g\cdot\frac{\Gamma-\gamma}{\text{Tnaught}}}}$$

$$\text{period} := 180 \qquad \gamma := \Gamma - \frac{4\cdot\pi^2\cdot\text{Tnaught}}{\text{period}^2\cdot g} \qquad \gamma = -0.024 \; {}^{o}C/m.$$

References:

C. F. Bohren, B. A. Albrecht: "*Atmospheric Thermodynamics*" pg 108-115
R. R. Rogers, M. K. Yau: "*A Short Course in Cloud Physics*" pg 29-32
J. P. Peixoto, A. H. Oort: "*Physics of Climate*" pg 139-142
D. Brunt, Quaterly Journal of the Royal Meteorological Society, 53, pg 30-32, 1927