Environmental Physics Computer Lab #7
Deterministic Chaos

Since Newton's time physicists have believed that it is possible to predict accurately the time evolution of any complex system by solving the set of differential equations representing Newton's second law applied to the componets of that system. For example, it is possible to predict the motion of the planets, moons and comets (e.g. Halley's) over a period of many years. It seemed that Newton's laws and his calculus can in principle be used to predict the future of a complex system. We now know this is not quite true because even relatively simple systems exhibit *deterministic chaos*. Indeed a system as simple as three objects interacting through gravitational forces (e. g. Sun and two planets) has complex and unpredictable dynamics. This was first noted by the French mathematician Poincare in 1892 in *"Les Methodes Nouvelles de la Mecanique Celeste"* .

An important example of deterministic chaos is the weather. The forecasts are never for more than a few days into the future. This is **not** because our models of the weather are not complete. More powerful supercomputers and even more complicated models will not help much with the long term weather forecasts. Edward Lorenz, an MIT meteorologist, showed that this is the case in a paper titled *"Deterministic Nonperiodic Flow"* published in 1963 in the Journal of Atmospheric Science. The weather is chaotic. Even with a perfect model of the weather it is not possible to predict it very far into the future. The reason is the so-called *butterfly effect*, a name coined by Lorenz. Let us assume that we have a good model of the weather. To use it to predict the future weather conditions we have to know the *initial conditions, i.e.* the weather conditions today. Lorenz showed that (for a particular model of atmospheric convection) two initial conditions different by very little lead to two very different long-term predictions.

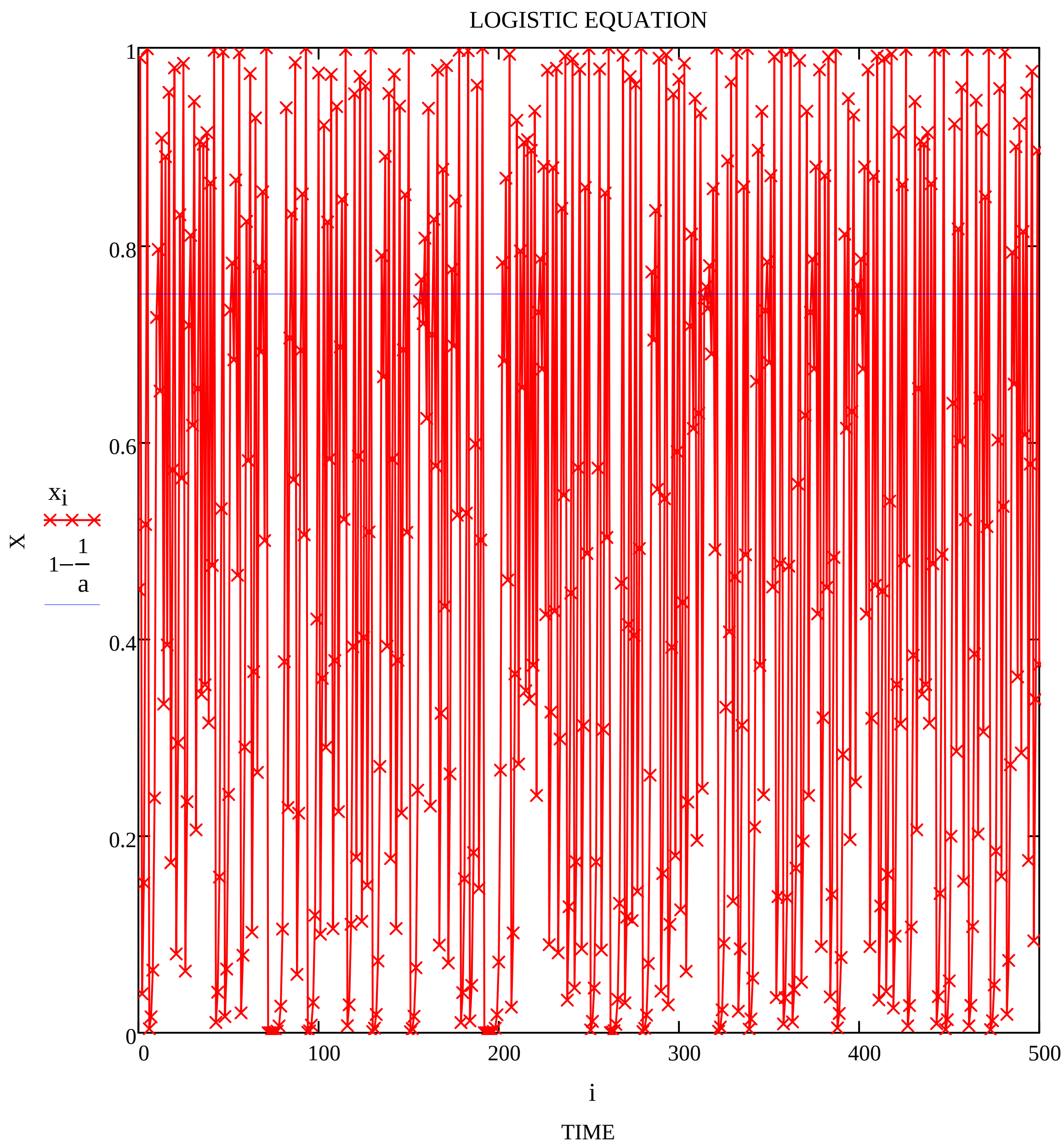
Another influential model exhibiting chaos was studied by Robert May, Nature, 261, 459 (1976) the context of population dynamics. This model describes the evolution in time of the population some species. The model is based on the following difference equation $x_{i+1}=ax_i(1-x_i)$, called the **logistic equation**. This model has been used extensively to study the dynamics of populations. I apparent simplicity is deceptive as this equation can predict highly complex dynamics exhibited by populations. x is a scaled variable taking values in the [0,1] interval and which is proportional to actual population. The index i measures the time in some unit, e.g. year. **a** is a constant which determines the type of time evolution, e.g. chaotic or periodic. The logistic equation is the simple **nonlinear difference** equation. It is the simplest generalization of the linear equation $x_{i+1}=ax_i$, which was studied early in the course as the exponential growth model of consumption. Within t latter context the addition of the nonlinear contribution $-x^2$ models the reduction in the rate of gro expected when the resources become scarce. **Nonlinearity** is a necessary ingredient of chaos.

We start now the simulations of the logistic equation. The number of iterations is N. The **initial condition** is the value for x_0. The parameter a determines the type of dynamics. We will iterate the logistic equation for different values of a, and the same initial condition a = 0.5, 1., 1.25, 2.25, 3., 3.4, 3.5, 3.55, 3.6, 3.63, 3.7, 3.8, 3.9, 4, all for x_0 = 0.45. Save the graphs of x vs i for all the values of a.

$N := 500$ $x_0 := 0.45$

$a := 4$ $i := 0 .. N$

$x_{i+1} := a \cdot x_i \cdot (1 - x_i)$



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QUALITATIVE CONCLUSIONS ON THE SIMULATIONS OF THE LOGISTIC MODEL

For a = 0.5 after many iterations x approaches zero, i.e. the population becomes extinct. x = 0 is called a stable fixed point.

For a = 1.25, 2.25, x approaches a nonzero value, i.e. the population is constant. The population reaches a sustainable level.

For a = 3.4 after a sufficient number of iterations x oscillates between two different values. This is stable situation, called period 2 cycle.

For a = 3.5 we have a stable situation where the population oscillates between four different values This is a period 4 cycle.

For a = 3.55 the population oscillates between eight values, i.e. period 8 cycle.

For a = 3.63 the stable population oscillates between six values, i.e. period 6 cycle.

For a = 3.6, 3.7, 3.8, 3.9, 4. the population dynamics is chaotic. No matter how many iterations the population never reaches a stable value or a cycle of a few values. The dependence of x on i looks random and unpredictable though it is produced by using a deterministic equation.
In the next computer session we will verify the butterfly effect (sensitivity to initial conditions).

To understand the long term behavior of x, we determine the location and stability of the fixed points. The fixed point equation: $x^* = ax^*(1-x^*)$ has two solutions $x^* = 0$ and $x^* = 1-1/a$. The slope of the map at the second fixed point is 2-a. One can see that the magnitude of the slope is larger than 1 for $a > 3$. Thus this fixed point is stable for $a < 3$ and unstable for $a > 3$. At the first fixed point ($x^* = 0$) the map slope is a. So that fixed point is stable for $a < 1$ and is unstable for $a > 1$.
Hence: for $a < 1$, x approaches 0 (extinction); for $1 < a < 3$, x approaches $1-1/a$ (sustainable).