Environmental Physics Computer Lab #8: The Butterfly Effect

The butterfly effect is the name coined by Lorenz for the main feature of deterministic chaos: sensitivity to initial conditions. Lorenz has presented his ideas in a talk given in 1972 at a meeting of the American Association for the Advancement of Science. I quote: *"... I am proposing that over years minuscule disturbances neither increase nor decrease the frequency of occurence of various weather events such as tornadoes; the most they may do is to modify the sequence in which these events occur. ...two particular weather situations differing by as little as the immediate influence of a single butterfly will generally after sufficient time evolve into two situations differing by as much as the presence of a tornado. ...The connection between this question and our ability to predict the weather is evident. Since we do not know exactly how many butterflies there are, nor where they are located, let alone which ones are flapping their wings at any instant, we cannot... accurately predict the occurence of tornados at a sufficiently distant time."*

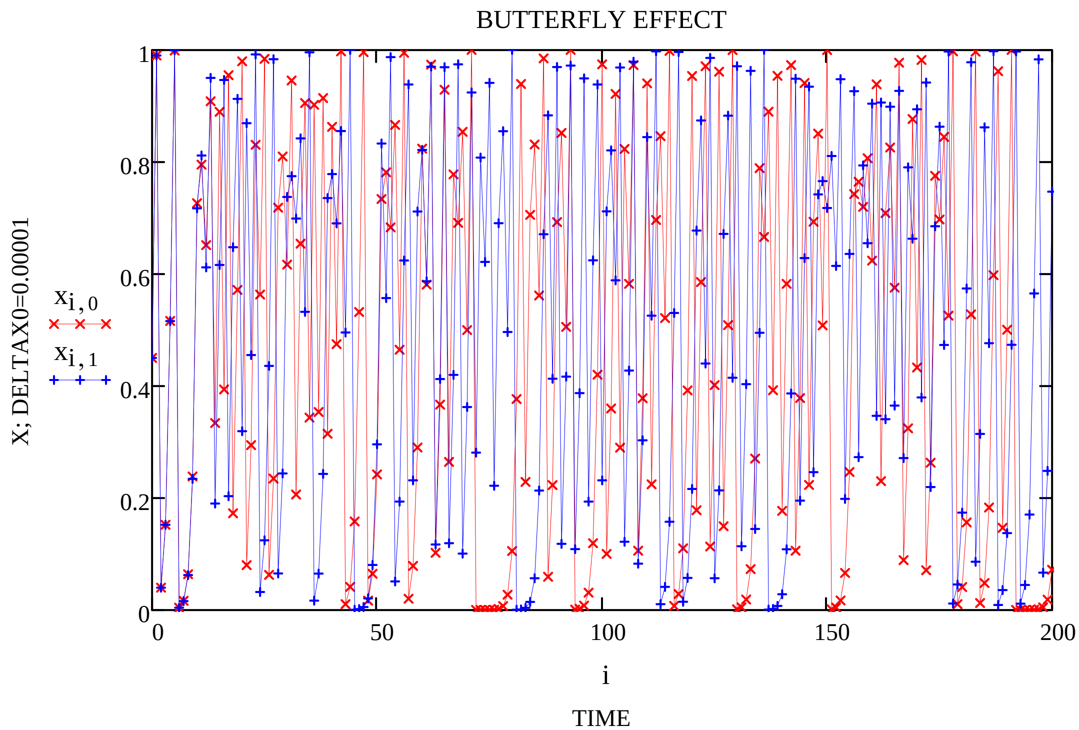
We will simulate next the logistic equation in order to visualise and understand the *butterfly effect*. We will achieve this by graphing x vs. i for two initial values that are slightly different. To better visualise the effect of the slight difference in the initial conditions we will also graph the difference of the two x vs i. Do this for: a = 0.5, 1., 1.25, 2.25, 3., 3.4, 3.5, 3.55, 3.6, 3.63, 3.7, 3.8, 3.9, 4. Take the initial conditions to be x_0 = 0.45 and 0.45 + 0.000001. Save all the simulations to your file.

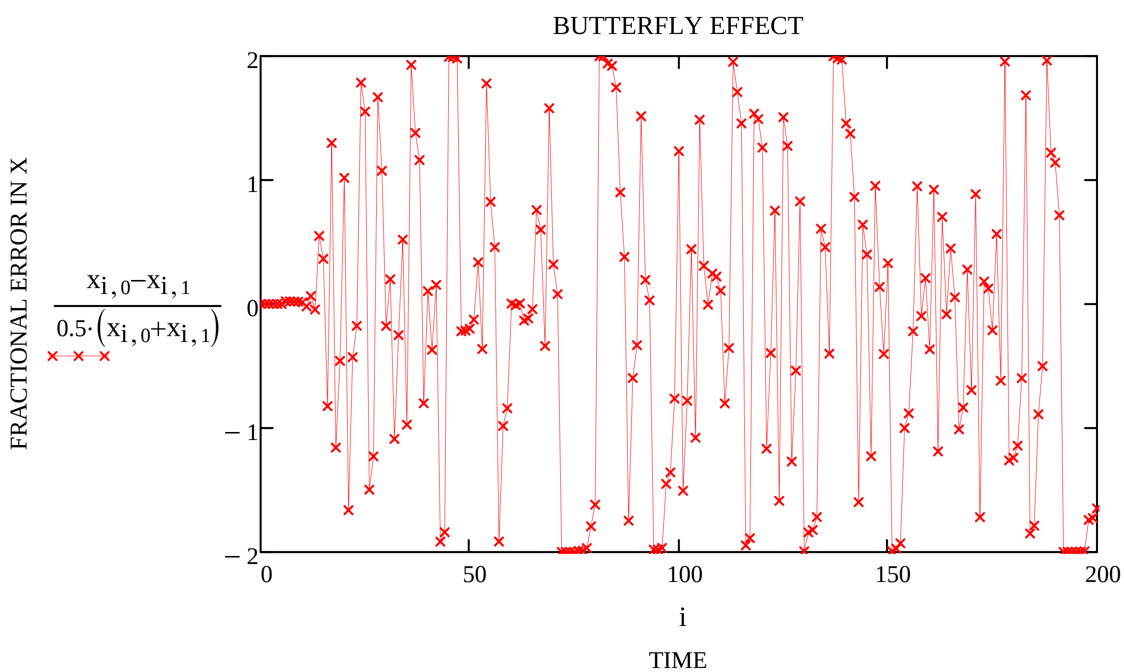
$a := 4$ $N := 200$

$j := 0..1$ $i := 0..N$

$x_{0,j} := 0.45 + j \cdot 0.00001$

$x_{i+1,j} := a \cdot x_{i,j} \cdot \left(1 - x_{i,j}\right)$





QUALITATIVE CONCLUSIONS ON THE SIMULATIONS OF THE LOGISTIC MODEL

For a = 0.5, 1.25, 2.25, 3.4, 3.5, 3.55, 3.63 the two curves from slightly different initial conditions are very close to each other. The dynamics is predictable.

For a = 3.6, 3.7, 3.8, 3.9, 4.0 the two curves from slightly different initial conditions become distinct after a finite number of iterations. The dynamics is chaotic and unpredictable, showing extreme sensitivity to initial conditions.

Physicists go beyond qualitative descriptions of nature by attempting to also to obtain quantitative characterisations of their experiments. In these simulations we have seen chaotic and periodic dynamics. Next step is to quantify these concepts by asking how much chaos and how much stability. An answer to this question is provided by the Lyapunov exponent named for a Russian mathematician who studied the stability of differential equations. The Lyapunov exponent is computed by averaging the logarithm of the slope of the map for a large number of iterations $\Lambda = (1/N)\Sigma\ln(dx_{i+1}/dx_i)$. Chaos occurs for a positive value of the Lyapunov exponent, and periodic dynamics occur for a negative value. Calculate the Lyapunov exponent for a = 2.5 and for a = 4.0 to verify this statement.

$a := 4 \qquad N := 5000$

$j := 0..1 \qquad i := 0..N-1$

$x_{0,j} := 0.45 + j \cdot 0.000001$

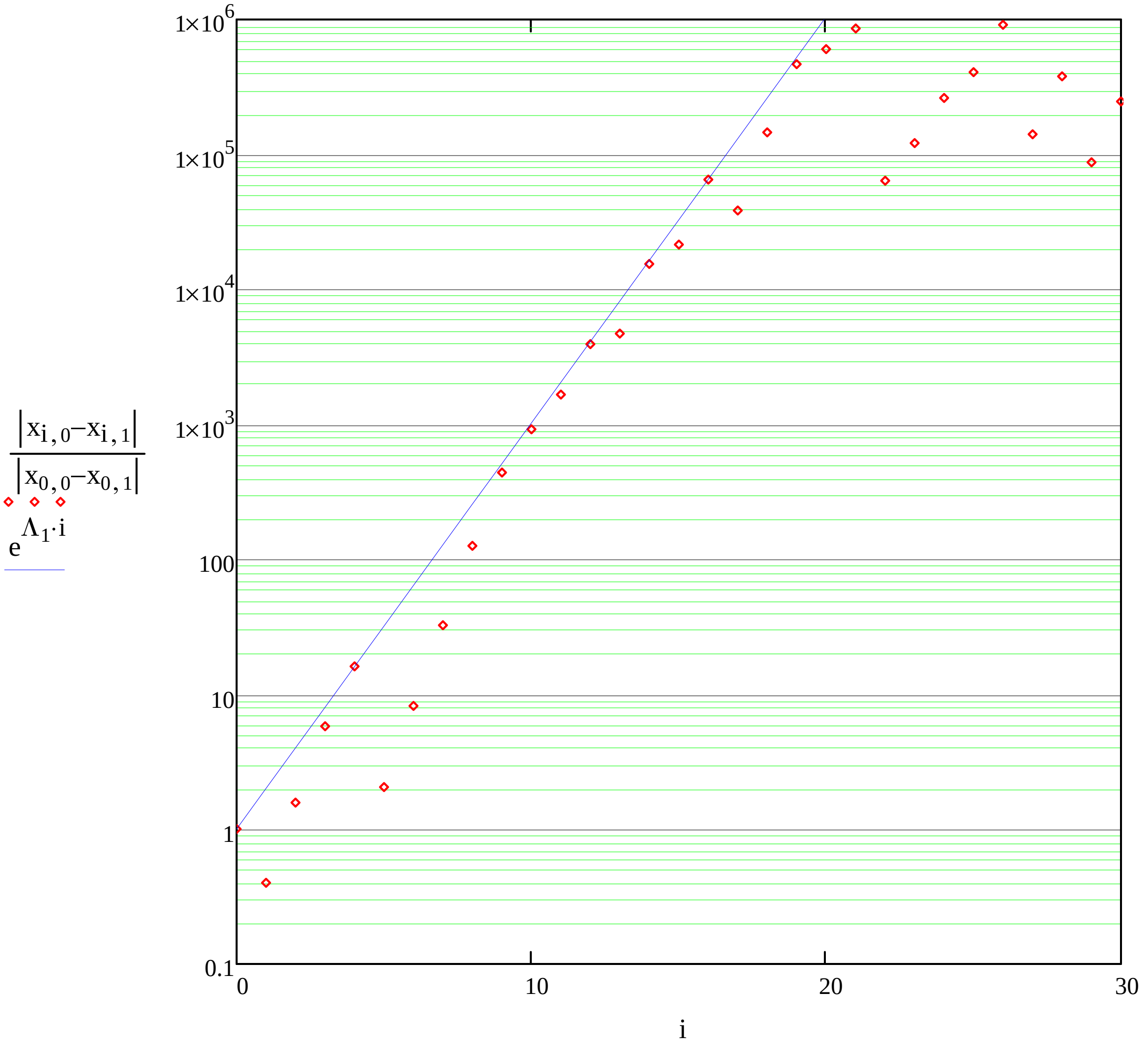
$x_{i+1,j} := a \cdot x_{i,j} \cdot (1 - x_{i,j})$

$$\Lambda_j := \frac{1}{N} \cdot \sum_{i=0}^{N-1} \ln\left[\left|a \cdot (1 - 2 \cdot x_{i,j})\right|\right]$$

$$\Lambda_j = \begin{pmatrix} 0.693 \\ 0.693 \end{pmatrix}$$

The absolute value of the distance between the two iterates $|x_{i,0} - x_{i,1}|$ varies approximately like exp(Λ*i) where Λ is the Lyapunov exponent. If the Lyapunov exponent is positive the distance becomes large (instability, chaos) while if the Lyapunov exponent is negative the distance diminishes to zero (stability, predictability). Illustrate this statement for a = 2.5 and 4.

$i := 0 .. 30$ $a = 4$



Why do you think the increase in the difference between the two x's saturates after a large enough number of iterations?

Next we will determine the dependence of the Lyapunov exponent on the parameter a. This calculation is memory intensive since to get accurate estimates of the exponent one needs many iterations (thousands) and also one needs as many values of a as possible because the dependence Λ(a) is non-monotonic and quite complex.

$M := 1000$

$j := 0 .. 2000$

$a_j := 3. + j \cdot 0.0005$

$i := 0 .. M - 1$

$x_{0,j} := 0.45$

$x_{i+1,j} := a_j \cdot x_{i,j} \cdot (1 - x_{i,j})$

$$\Lambda_j := \frac{1}{M} \cdot \sum_{i=0}^{M-1} \ln\left[\left|a_j \cdot (1 - 2 \cdot x_{i,j})\right|\right]$$