Electricity and Magnetism
Computer Lab #1: Vector algebra and calculus

We are going to learn how to use MathCad. First we use MathCad as a calculator. Type: 54.4+3.2-(3/5+4/5)^2= To get the power we type hat.

$$54.4 + 3.2 - \left(\frac{3}{5} + \frac{4}{5}\right)^2 = 55.64$$

To get square root type \. For instance type: \2/3 space bar+3.1^2=

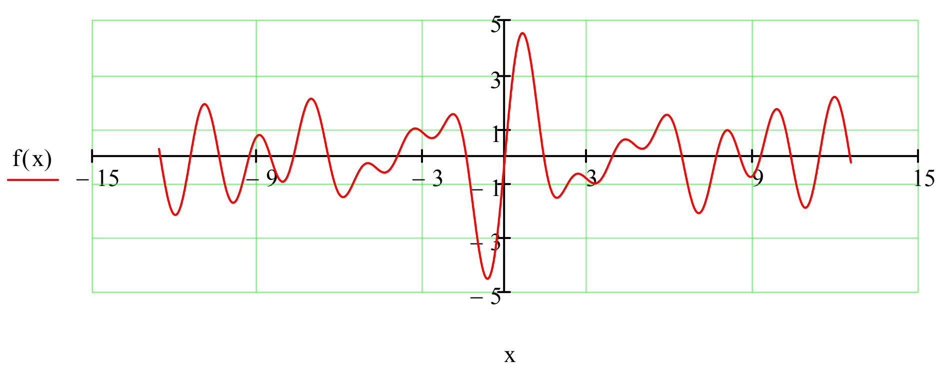
$$\sqrt{\frac{2}{3} + 3.1^2} = 3.206$$

Next we learn how to define functions and variables. An important operator is the *assign* operator. Type *colon*. It shows as := Another important operator is *until*. Type *semi-colon*. It shows ..

$$f(x) := \sin(\sqrt{2}\cdot x) + \sin(\sqrt{3}\cdot x) + \sin(\sqrt{5}\cdot x) + \sin(\sqrt{7}\cdot x) + \sin(\sqrt{11}\cdot x)$$

$$x := -4\cdot\pi, -4\cdot\pi + .01 \,..\, 4\cdot\pi$$

To type greek letters, e. g. π, you either use the Greek Symbol Toolbar or type the corresponding English letter followed by *control g*.
To graph the function defined above click on the graph palette and choose xy plot. Then click on Format Graph and adjust the graph format to the desired form.



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We draw a sphere using a 3D parametric graph. We use spherical coordinates.

$x(r,\theta,\phi) := r\cdot\cos(\phi)\cdot\sin(\theta)$

$y(r,\theta,\phi) := r\cdot\sin(\phi)\cdot\sin(\theta)$

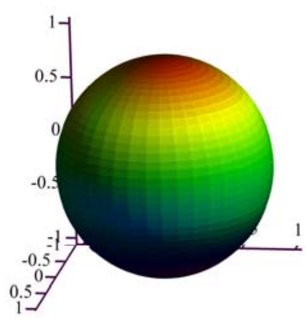
$z(r,\theta,\phi) := r\cdot\cos(\theta)$

$i := 0..40$ $\qquad j := 0..40$

$r := 1$ $\qquad \theta_i := \pi\cdot\frac{i}{40}$ $\qquad \phi_j := 2\pi\cdot\frac{j}{40}$

$X_{i,j} := x(r,\theta_i,\phi_j)$ $\qquad Y_{i,j} := y(r,\theta_i,\phi_j)$ $\qquad Z_{i,j} := z(r,\theta_i,\phi_j)$

Now chose Graph Surface Plot. In the place holder type (X,Y,Z).



(X, Y, Z)

Beside assign (definition) := and evaluate numerically = operators, we can also use evalua
symbolicaly (from evaluation toolbar).

Compute volume of sphere: $\int_0^{\pi}\int_0^{2\cdot\pi}\int_0^{R} r^2\cdot\sin(\theta)\,dr\,d\phi\,d\theta \rightarrow \frac{4\cdot\pi\cdot R^3}{3}$

Compute area of sphere: $\int_0^{\pi}\int_0^{2\cdot\pi} R^2\cdot\sin(\theta)\,d\phi\,d\theta \rightarrow 4\cdot\pi\cdot R^2$

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We graph next a cylinder, by using cylindrical coordinates.

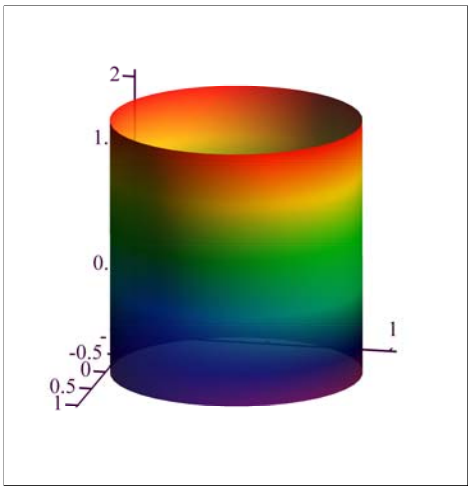
$s := 1$

$x(s,\phi,\zeta) := s\cdot\cos(\phi)$ $\quad y(s,\phi,\zeta) := s\cdot\sin(\phi)$ $\quad z(s,\phi,\zeta) := \zeta$

$i := 0..40$ $\quad j := 0..40$

$\phi_i := 2\pi\cdot\frac{i}{40}$ $\quad \zeta_j := \frac{j}{20}$

$X_{i,j} := x(s,\phi_i,\zeta_j)$ $\quad Y_{i,j} := y(s,\phi_i,\zeta_j)$ $\quad Z_{i,j} := z(s,\phi_i,\zeta_j)$



(X, Y, Z)

Compute volume of cylinder: $\int_0^H \int_0^{2\cdot\pi} \int_0^R s\,ds\,d\phi\,dz \rightarrow \pi\cdot H\cdot R^2$

Compute lateral area of cylinder: $\int_0^H \int_0^{2\cdot\pi} R\,d\phi\,dz \rightarrow 2\cdot\pi\cdot H\cdot R$

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Creating a vector: Use the insert matrix command.

$$V1 := \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad V2 := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Dot product: $V1 \cdot V2 = 1$

Note that the dot product does not work with vectors defined as rows:

$$V3 := (1 \quad 2) \qquad V4 := (2 \quad 4)$$

$$V3 \cdot V4 = \blacksquare$$

Cross product:

$$Z := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad Y := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad Z \times Y = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

The cross product does not work unless the vectors have three elements:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \blacksquare$$

Transpose:

$$Z^T = (0 \quad 0 \quad 1) \qquad Y^T = (0 \quad 1 \quad 0)$$

$$Z^T \times Y^T = \blacksquare$$

$$Z^T \cdot Y^T = \blacksquare$$

Note that the cross and dot products do not work with vectors defined as rows.

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Matrices, Determinants:

$$M1 := \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 2 & 3 \end{pmatrix} \qquad M1^T = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ -1 & -2 & 3 \end{pmatrix}$$

$$|M1| = 0 \qquad |M1^T| = 0$$

$M1^{-1} = ▮$ Note that if determinant of matrix is zero there is no inverse.

$$M2 := \begin{pmatrix} 1 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 3 \end{pmatrix} \qquad M2^T = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 0 & 2 \\ -1 & -2 & 3 \end{pmatrix}$$

$$|M2| = 16 \qquad |M2^T| = 16$$

Note the determinant of transpose matrix equals determinant of original matrix.

$$M2^{-1} = \begin{pmatrix} 0.25 & -0.5 & -0.25 \\ 0.25 & 0.25 & 0.25 \\ -0.25 & 0 & 0.25 \end{pmatrix}$$

$$M2 \cdot M2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M2^{-1} \cdot M2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is identity matrix.

$$|M2^{-1}| = 0.063 \qquad |M2| \cdot |M2^{-1}| = 1$$

$$M3 := \begin{pmatrix} 1 & 2 & 3 \\ -3 & -1 & -2 \\ 2 & -5 & 1 \end{pmatrix} \qquad M3 \cdot M2 = \begin{pmatrix} 0 & 8 & 4 \\ -3 & -10 & -1 \\ 13 & 6 & 11 \end{pmatrix} \qquad M2 \cdot M3 = \begin{pmatrix} -7 & 5 & -2 \\ -6 & 6 & -8 \\ 1 & -15 & 2 \end{pmatrix}$$

Matrix product is not commutative.

$$|M3 \cdot M2| = 608 \qquad |M2| \cdot |M3| = 608$$

Determinant of product equals product of determinants.

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Vector Triple Product:

$$A := \begin{pmatrix} 3 \\ 2 \\ -3.5 \end{pmatrix} \qquad B := \begin{pmatrix} 2.3 \\ -3.2 \\ -5.3 \end{pmatrix} \qquad C := \begin{pmatrix} -1.5 \\ 2.5 \\ .5 \end{pmatrix}$$

We verify **A**x(**B**x**C**)= (**A*****C**)**B**-(**A*****B**)**C**

$$B \times C = \begin{pmatrix} 11.65 \\ 6.8 \\ 0.95 \end{pmatrix} \qquad A \times (B \times C) = \begin{pmatrix} 25.7 \\ -43.625 \\ -2.9 \end{pmatrix}$$

$A \cdot B = 19.05$

$A \cdot C = -1.25$

$$(A \cdot C) \cdot B - (A \cdot B) \cdot C = \begin{pmatrix} 25.7 \\ -43.625 \\ -2.9 \end{pmatrix}$$

We verify **(A**x**B)**x**C** = (**A*****C**)**B** - (**B*****C**)**A**

$$A \times B = \begin{pmatrix} -21.8 \\ 7.85 \\ -14.2 \end{pmatrix} \qquad (A \times B) \times C = \begin{pmatrix} 39.425 \\ 32.2 \\ -42.725 \end{pmatrix}$$

$A \cdot C = -1.25$

$B \cdot C = -14.1$

$$(A \cdot C)B - (B \cdot C) \cdot A = \begin{pmatrix} 39.425 \\ 32.2 \\ -42.725 \end{pmatrix}$$

Scalar Triple Product: Equals the volume of the parallelipiped generated by A, B, C.

We verify **A***(**B**x**C**) = **C***(**A**x**B**) = **B***(**A**x**C**) = (**A**x**B**)***C**

$A \cdot (B \times C) = 45.225$ $\qquad C \cdot (A \times B) = 45.225$ $\qquad B \cdot (C \times A) = 45.225$

$(A \times B) \cdot C = 45.225$

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Gradient, Divergence, Curl, Laplacean:

$\nabla_{\blacksquare} \blacksquare$

To insert the gradient (del) operator, press [Ctrl] [Shift] G:

Type the variable vector in the lower placeholder, and type the function in the upper placeholder. To illustrate this we consider the potential of a point-like charge Q located at the origin. Up to a multiplicative constant Q/ε_0, it is:

$$V(R) := \frac{1}{\sqrt{(R_0)^2 + (R_1)^2 + (R_2)^2}}$$

To evaluate the gradient we need to use the evaluate symbolically operator (you find it in Evaluation Toolbar and in Symbolic Toolbar).

$$\nabla_R V(R) \rightarrow \begin{bmatrix} -\dfrac{R_0}{\left[(R_0)^2 + (R_1)^2 + (R_2)^2\right]^{\frac{3}{2}}} \\ -\dfrac{R_1}{\left[(R_0)^2 + (R_1)^2 + (R_2)^2\right]^{\frac{3}{2}}} \\ -\dfrac{R_2}{\left[(R_0)^2 + (R_1)^2 + (R_2)^2\right]^{\frac{3}{2}}} \end{bmatrix}$$

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The electric field is the negative of gradient of potential:

$$E(x,y,z) := \begin{bmatrix} \dfrac{x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \\ \dfrac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \\ \dfrac{z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \end{bmatrix}$$

The divergence operator ∇* and the curl operator ∇x are defined next.

$$\mathrm{div}(A,x,y,z) := \frac{d}{dx}A(x,y,z)_0 + \frac{d}{dy}A(x,y,z)_1 + \frac{d}{dz}A(x,y,z)_2$$

$$\mathrm{curl}(A,x,y,z) := \begin{pmatrix} \frac{d}{dy}A(x,y,z)_2 - \frac{d}{dz}A(x,y,z)_1 \\ \frac{d}{dz}A(x,y,z)_0 - \frac{d}{dx}A(x,y,z)_2 \\ \frac{d}{dx}A(x,y,z)_1 - \frac{d}{dy}A(x,y,z)_0 \end{pmatrix}$$

For the electrical field **E** of a point charge at location away from charge location: the Gauss law is ∇***E**=0 and the Faraday law is ∇x**E** = 0. Also the potential V, **E** = - ∇V, satisfies the Laplace equation $\nabla^2 V = 0$. We verify this next.

$$\mathrm{div}(E,xx,yy,zz) \text{ simplify } \rightarrow 0 \qquad \mathrm{curl}(E,xx,yy,zz) \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{laplacean}(F,x,y,z) := \frac{d^2}{dx^2}F(x,y,z) + \frac{d^2}{dy^2}F(x,y,z) + \frac{d^2}{dz^2}F(x,y,z)$$

$$V(x,y,z) := \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\mathrm{laplacean}(V,xx,yy,zz) \text{ simplify } \rightarrow 0$$

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By using vector field graph we can visualize the lines of electric field.

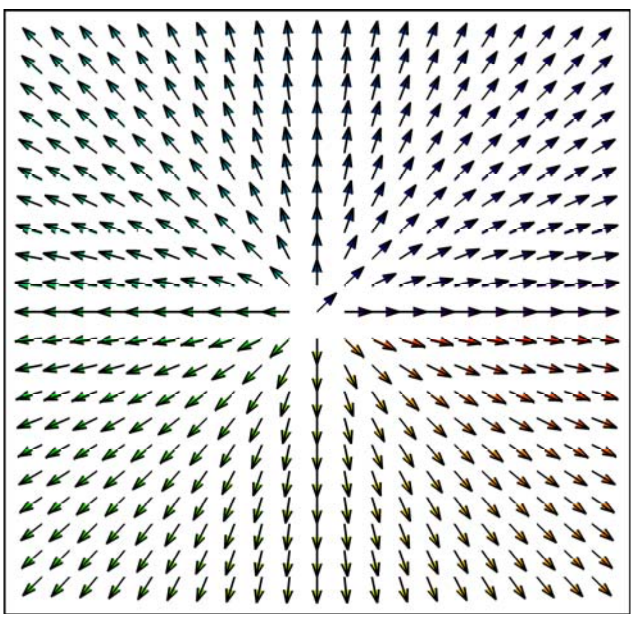
$$Ex(x,y,z) := \frac{x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \qquad Ey(x,y,z) := \frac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \qquad Ez(x,y,z) := \frac{z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

$i := 0..20 \qquad j := 0..20$

$$X_{i,j} := \frac{i}{10} - 1. + 10^{-6} \qquad Y_{i,j} := \frac{j}{10} - 1. + 10^{-6}$$

The reason we add the 10^{-6} to the coordinates X and Y is to evaluate the field away from the origin (where the charged particle is located) where E is infinite.

$$ex_{i,j} := \frac{Ex\left(X_{i,j},Y_{i,j},0\right)}{\sqrt{Ex\left(X_{i,j},Y_{i,j},0\right)^2 + Ey\left(X_{i,j},Y_{i,j},0\right)^2}} \qquad ey_{i,j} := \frac{Ey\left(X_{i,j},Y_{i,j},0\right)}{\sqrt{Ex\left(X_{i,j},Y_{i,j},0\right)^2 + Ey\left(X_{i,j},Y_{i,j},0\right)^2}}$$



(ex, ey)

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Griffiths Ch.1 Problem 11

Calculate the gradients of the following functions.

$$\nabla_{xx,yy,zz}\left(xx^2 + yy^3 + zz^4\right) \rightarrow \begin{pmatrix} 2\cdot xx \\ 3\cdot yy^2 \\ 4\cdot zz^3 \end{pmatrix}$$

$$\nabla_{xx,yy,zz}\left(xx^2\cdot yy^3\cdot zz^4\right) \rightarrow \begin{pmatrix} 2\cdot xx\cdot yy^3\cdot zz^4 \\ 3\cdot xx^2\cdot yy^2\cdot zz^4 \\ 4\cdot xx^2\cdot yy^3\cdot zz^3 \end{pmatrix}$$

$$\nabla_{xx,yy,zz}\left(e^{xx}\cdot \sin(yy)\cdot \ln(zz)\right) \rightarrow \begin{pmatrix} e^{xx}\cdot \sin(yy)\cdot \ln(zz) \\ e^{xx}\cdot \cos(yy)\cdot \ln(zz) \\ \dfrac{e^{xx}\cdot \sin(yy)}{zz} \end{pmatrix}$$

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Griffiths Ch.1 Problems 15, 18

Calculate divergence and curl of the following functions.

$$va(x,y,z) := \begin{pmatrix} x^2 \\ 3\cdot x\cdot z^2 \\ -2\cdot x\cdot z \end{pmatrix} \qquad div(va,xx,yy,zz) \to 0 \qquad curl(va,xx,yy,zz) \to \begin{pmatrix} -6\cdot xx\cdot zz \\ 2\cdot zz \\ 3\cdot zz^2 \end{pmatrix}$$

$$vb(x,y,z) := \begin{pmatrix} x\cdot y \\ 2\cdot y\cdot z \\ 3\cdot z\cdot x \end{pmatrix} \qquad div(vb,xx,yy,zz) \to 3\cdot xx + yy + 2\cdot zz \qquad curl(vb,xx,yy,zz) \to \begin{pmatrix} -2\cdot yy \\ -3\cdot zz \\ -xx \end{pmatrix}$$

$$vc(x,y,z) := \begin{pmatrix} y^2 \\ 2\cdot x\cdot y + z^2 \\ 2\cdot y\cdot z \end{pmatrix} \qquad div(vc,xx,yy,zz) \to 2\cdot xx + 2\cdot yy \qquad curl(vc,xx,yy,zz) \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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