Lecture Notes 11

Bose-Einstein Condensation

In Lecture 10 we obtained the following formulas for N and U:

$$N = \frac{gV}{{\lambda_T}^3} F_{3/2}(\xi) \tag{1}$$

$$U = \frac{gV}{{\lambda_T}^3} \frac{3}{2} k_B T F_{5/2}(\xi) \tag{2}$$

The thermal (de Broglie) wavelength is: $\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$ and

$$F_m(\xi) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{x^{m-1}}{e^x \xi^{-1} - 1} dx \tag{3}$$

Since U = (3/2)pV we get from Eq.(2):

$$p = \frac{g k_B T}{{\lambda_T}^3} F_{5/2}(\xi) \tag{4}$$

Einstein observed in 1925 that if the number of particles is conserved, the ideal gas will undergo a phase transition at low enough temperatures. We fix the number of bosons per volume N/gV, while lowering temperature and thus increasing the thermal wavelength. In Figure 1 we show $N{\lambda_T}^3/gV$ vs fugacity ξ:

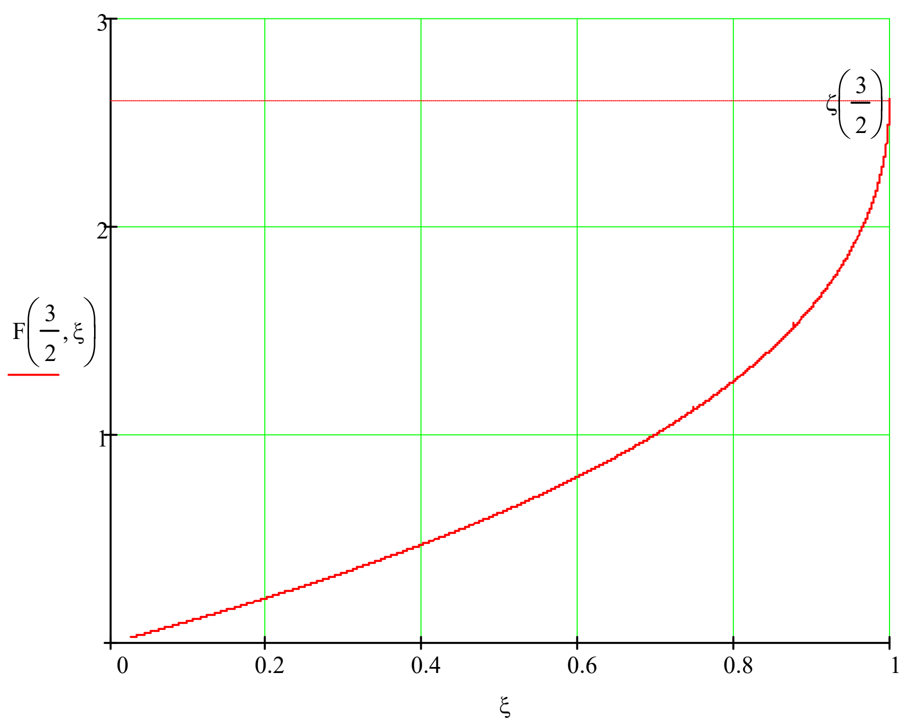

Figure 1: $(N/gV){\lambda_T}^3 = F(3/2, \xi)$ vs ξ

For $N\lambda_T^3/gV < \zeta(3/2) = 2.612$ the fugacity is $\xi < 1$. For $N\lambda_T^3/gV > \zeta(3/2)$ the fugacity is equal to unity $\xi = 1$, or the chemical potential equals zero $\mu = 0$. At these low temperatures the gas condenses in the ground state: n_0/V is finite. Equation 1 is modified to fix this breakdown of the integral approximation:

$$\frac{N-n_0}{V} = \frac{g}{\lambda_T{}^3} F_{3/2}(1) \tag{5}$$

At the condensation temperature:

$$\frac{N}{V} = \frac{g}{\lambda_{TC}{}^3} F_{3/2}(1) = \frac{g}{\lambda_{TC}{}^3} \zeta(3/2) \tag{6}$$

Divide Eq. (5) by Eq. (6):

$$\frac{N-n_0}{N} = \frac{\lambda_{TC}{}^3}{\lambda_T{}^3} \tag{7}$$

Using the definition for thermal wavelength:

$$\frac{n_0}{N} = 1 - (\frac{T}{T_C})^{3/2} \tag{8}$$

Hence at T = 0K: $n_0/N = 1$ because all bosons are in the ground state. At $T = T_C$: $n_0/N = 0$ as the fraction of all bosons in any state is negligible.

The phase diagram in the temperature, pressure plane is obtained by substituting $\xi = 1$and $T = T_C$ in Eq. (4):

$$p = \frac{gk_BT_C}{\lambda_{TC}{}^3} F_{5/2}(1) = \frac{gk_BT_C}{\lambda_{TC}{}^3} \zeta(5/2) \tag{9}$$

where $\zeta(5/2) = 1.341$.

In the condensate, $T \leq T_C$, the energy is obtained by substituting $\xi =1$ in Equation (2):

$$U = \frac{gV}{\lambda_T{}^3}\frac{3}{2} k_BTF_{5/2}(1) = \frac{gV}{\lambda_T{}^3}\frac{3}{2} k_BT\zeta(5/2) \tag{10}$$

We calculate energy per boson by dividing Eq. (10) by Eq. (6):

$$\frac{U}{N} = \frac{3}{2} k_BT \frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} (\frac{T}{T_C})^{3/2} \tag{11}$$

The isochoric heat capacity is obtained by differentiating Eq. (11) at fixed V and N:

$$C_V = \frac{\partial U}{\partial T} = 1.925Nk_B(\frac{T}{T_C})^{3/2} \tag{12}$$

Note that at the condensation temperature $C_V = 1.925Nk_B > 1.5Nk_B$, the high temperature, classical, heat capacity.

Appendix

We study the special functions $F_m(\xi)$ introduced in Eq.(12).

$$F_m(\xi) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{x^{m-1}}{e^{x}\xi^{-1}-1} dx \qquad (23)$$

We can rewrite Eq (23) as:

$$F_m(\xi) = \sum_{n=1}^{\infty} \frac{\xi^n}{n^m} \qquad (26)$$

In the classical limit, $\xi << 1$, and $F_m(\xi) \simeq \xi$, as we have already shown to be the case in Eq.(19).

In the quantum limit, where Bose-Einstein condensation occurs, $\xi = 1$:

$$F_m(1) = \sum_{n=1}^{\infty} \frac{1}{n^m} = \zeta(m) \qquad (27)$$

Here we introduced the Riemann zeta function $\zeta(m)$. The zeta function of real variable was used first by Euler. Here are a couple of values that are used in the study of Bose condensation: $\zeta(3/2) = 2.612375$, $\zeta(5/2) = 1.341488$. The zeta function $\zeta(m)$ is finite for $m > 1$. It diverges as m approaches 1:

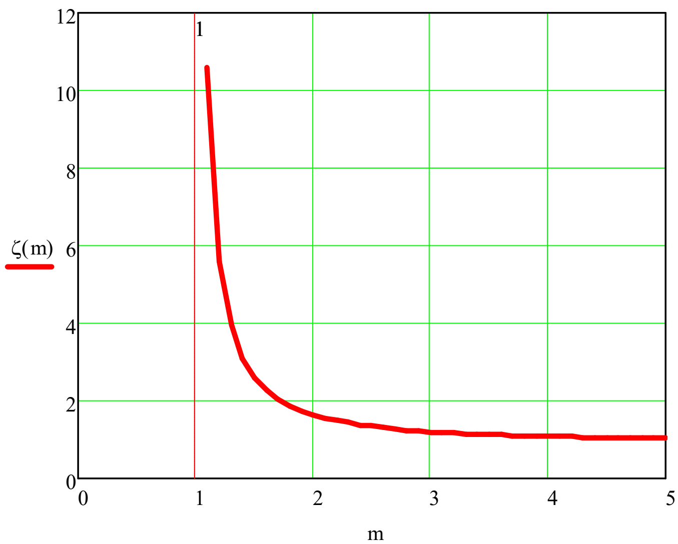

Figure 2: Riemann zeta function ζ(m) diverges as m approaches unity from above.

<u>Readings</u>
Callen Ch. 18
Problem Set 10
Computer lab 6