## Contents

### Chapter I

**Examples of curves**

1. Introduction .............................................. 1  
2. The topology of a few specific plane curves .......... 5  
3. Intersecting curves ..................................... 19  
4. Curves over $\mathbb{Q}$ ................................... 25  

### Chapter II

**Plane curves**

1. Projective spaces ........................................ 28  
2. Affine and projective varieties; examples ............ 34  
3. Implicit mapping theorems .............................. 46  
4. Some local structure of plane curves ................. 54  
5. Sphere coverings ....................................... 66  
6. The dimension theorem for plane curves ............. 75  
7. A Jacobian criterion for nonsingularity .............. 80  
8. Curves in $\mathbb{P}^2(\mathbb{C})$ are connected ......... 86  
9. Algebraic curves are orientable ...................... 93  
10. The genus formula for nonsingular curves .......... 97  

### Chapter III

**Commutative ring theory and algebraic geometry** ....... 103  

1. Introduction ........................................... 103  
2. Some basic lattice-theoretic properties of varieties and ideals .... 106  
3. The Hilbert basis theorem ............................ 117  
4. Some basic decomposition theorems on ideals and varieties .... 121  

vii
5 The Nullstellensatz: Statement and consequences 124
6 Proof of the Nullstellensatz 128
7 Quotient rings and subvarieties 132
8 Isomorphic coordinate rings and varieties 136
9 Induced lattice properties of coordinate ring surjections; examples 143
10 Induced lattice properties of coordinate ring injections 150
11 Geometry of coordinate ring extensions 155

Chapter IV

Varieties of arbitrary dimension 163
1 Introduction 163
2 Dimension of arbitrary varieties 165
3 The dimension theorem 181
4 A Jacobian criterion for nonsingularity 187
5 Connectedness and orientability 191
6 Multiplicity 193
7 Bézout's theorem 207

Chapter V

Some elementary mathematics on curves 214
1 Introduction 214
2 Valuation rings 215
3 Local rings 235
4 A ring-theoretic characterization of nonsingularity 248
5 Ideal theory on a nonsingular curve 255
6 Some elementary function theory on a nonsingular curve 266
7 The Riemann-Roch theorem 279

Bibliography 297
Notation index 299
Subject index 301
CHAPTER I
Examples of curves

1 Introduction

The principal objects of study in algebraic geometry are algebraic varieties. In this introductory chapter, which is more informal in nature than those that follow, we shall define algebraic varieties and give some examples; we then give the reader an intuitive look at a few properties of a special class of varieties, the "complex algebraic curves." These curves are simpler to study than more general algebraic varieties, and many of their simply-stated properties suggest possible generalizations. Chapter II is essentially devoted to proving some of the properties of algebraic curves described in this chapter.

Definition 1.1. Let $k$ be any field.

(1.1.1) The set $\{(x_1, \ldots, x_n) | x_i \in k\}$ is called **affine $n$-space over $k$**; we denote it by $k^n$, or by $k_{x_1, \ldots, x_n}$. Each $n$-tuple of $k^n$ is called a **point**.

(1.1.2) Let $k[X_1, \ldots, X_n] = k[X]$ be the ring of polynomials in $n$ indeterminants $X_1, \ldots, X_n$, with coefficients in $k$. Let $p(X) \in k[X] \setminus k$. The set

$$V(p) = \{(x) \in k^n | p(x) = 0\}$$

is called a **hypersurface of $k^n$**, or an **affine hypersurface**.

(1.1.3) If $\{p_i(X)\}$ is any collection of polynomials in $k[X]$, the set

$$V(\{p_i\}) = \{(x) \in k^n | \text{each } p_i(x) = 0\}$$

is called an **algebraic variety in $k^n$**, and **affine algebraic variety**, or, if the context is clear, just a **variety**. If we wish to make explicit reference to the field $k$, we say **affine variety over $k$**, $k$-**variety**, etc.; $k$ is called the **ground field**. We also say $V(\{p_i\})$ is **defined by** $\{p_i\}$.
1: Examples of curves

(1.1.4) \( k^2 \) is called the affine plane. If \( p \in k[X_1, X_2] \setminus k \), \( V(p) \) is called a plane affine curve (or plane curve, affine curve, curve, etc., if the meaning is clear from context).

We will show later on, in Section III.3, that any variety can be defined by only finitely many polynomials \( p \).

Here are some examples of varieties in \( \mathbb{R}^2 \).

Example 1.2

(1.2.1) Any variety \( V(aX^2 + bXY + cY^2 + dX + eY + f) \) where \( a, \ldots, f \in \mathbb{R} \). Hence all circles, ellipses, parabolas, and hyperbolas are affine algebraic varieties; so also are all lines.

(1.2.2) The “cusp” curve \( V(Y^2 - X^2) \); see Figure 1.

(1.2.3) The “alpha” curve \( V(Y^2 - X^2(X + 1)) \); see Figure 2.

![Figure 1](image1)

![Figure 2](image2)

(1.2.4) The cubic \( V(Y^2 - X(X^2 - 1)) \); see Figure 3. This example shows that algebraic curves in \( \mathbb{R}^2 \) need not be connected.

(1.2.5) If \( V(p_1) \) and \( V(p_2) \) are varieties in \( \mathbb{R}^2 \), then so is \( V(p_1) \cup V(p_2) \); it is just \( V(p_1 \cdot p_2) \), as the reader can check directly from the definition. Hence one has a way of manufacturing all sorts of new varieties. For instance, \( (X^2 + Y^2 - 1)(X^2 + Y^2 - 4) = 0 \) defines the union of two concentric circles (Figure 4).

(1.2.6) The graph \( V(Y - p(X)) \) in \( \mathbb{R}^2 \) of any polynomial \( Y = p(X) \in \mathbb{R}[X] \) is also an algebraic variety.

(1.2.7) If \( p_1, p_2 \in \mathbb{R}[X, Y] \), then \( V(p_1, p_2) \) represents the simultaneous solution set of two polynomial equations. For instance, \( V(X, Y) = \{(0, 0)\} \subseteq \mathbb{R}^2 \), while \( V(X^2 + Y^2 - 1, X - Y) \) is the two-point set

\[
\left\{ \left( \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2} \right), \left( -\frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2} \right) \right\}
\]

in \( \mathbb{R}^2 \).
1: Introduction

(1.2.8) In $\mathbb{R}^3$, any conic is an algebraic variety, examples being the sphere $V(X^2 + Y^2 + Z^2 - 1)$, the cylinder $V(X^2 + Y^2 - 1)$, the hyperboloid $V(X^2 - Y^2 - Z^2 - 1)$, and so on. A circle in $\mathbb{R}^3$ is also a variety, being represented, for example, as $V(X^2 + Y^2 + Z^2 - 1, X)$ (geometrically the intersection of a sphere and the $(Y, Z)$-plane). Any point $(a, b, c)$ in $\mathbb{R}^3$ is the variety $V(X - a, Y - b, Z - c)$ (geometrically, the intersection of the three planes $X = a$, $Y = b$, and $Z = c$).

Now suppose (still using $k = \mathbb{R}$) that we have written down a large number of sets of polynomials, and that we have sketched their corresponding varieties in $\mathbb{R}^n$. It is quite natural to look for some regularity. How do algebraic varieties behave? What are their basic properties?

First, perhaps a simple "dimensionality property" might suggest itself. For our immediate purposes, we may say that $V \subset \mathbb{R}^n$ has dimension $d$ if $V$ contains a homeomorph of $\mathbb{R}^d$, and if $V$ is the disjoint union of finitely many homeomorphs of $\mathbb{R}^d$ ($i \leq d$). Then in all examples given so far, each equation introduces one restriction on the dimension, so that each variety defined by one equation has dimension one less than the surrounding space—i.e., the variety has codimension 1. (In $k^n$, "codimension" means "$n -$ dimension.") And each variety defined by two (essentially different) equations has dimension two less than the surrounding (or "ambient") space (codimension 2), etc. Hence the sphere $V(X^2 + Y^2 + Z^2 - 1)$ in $\mathbb{R}^3$ has dimension $3 - 1 = 2$, the circle $V(X^2 + Y^2 + Z^2 - 1, X)$ in $\mathbb{R}^3$ has dimension $3 - 2 = 1$, and the point $V(X - a, Y - b, Z - c)$ in $\mathbb{R}^3$ has dimension $3 - 3 = 0$. This same thing happens in $\mathbb{R}^n$ with homogeneous linear equations—each new linearly independent equation cuts down the dimension of the resulting subspace by one.

But if we look down our hypothetical list a bit further, we come to the polynomial $X^2 + Y^2$; $X^2 + Y^2$ defines only the $Z$-axis in $\mathbb{R}^3$. This one equation cuts down the dimension of $\mathbb{R}^3$ by two—that is, the $Z$-axis has co-dimension two in $\mathbb{R}^3$. And further down the list we see $X^2 + Y^2 + Z^2$; the
1: Examples of curves

associated variety is only the origin in \( \mathbb{R}^3 \). And if this is not bad enough, 
\( X^2 + Y^2 + Z^2 + 1 \) defines the empty set \( \emptyset \) in \( \mathbb{R}^3 \)! Clearly then, one 
equation does not always cut down the dimension by one.

We might try simply restricting our attention to the “good” sets of polynomials, where the hoped-for dimensional property holds. But one “good” polynomial together with another one may not yield a “good” set of polynomials. For instance, two spheres in \( \mathbb{R}^3 \) may not intersect in a circle (codimension 2), but rather in a point, or in the empty set.

Though things might not look very promising at this point, mathematicians have often found their way out of similar situations. For instance, mathematicians of antiquity thought that only certain nonconstant polynomials in \( \mathbb{R}[X] \) had zeros. But the exceptional status of polynomials having only real roots was removed once the field \( \mathbb{R} \) was extended to its algebraic completion, \( \mathbb{C} \) = field of complex numbers. One then had a most beautiful and central result, the fundamental theorem of algebra. (Every nonconstant polynomial \( p(X) \in \mathbb{C}[X] \) has a zero, and the number of these zeros, when counted with multiplicity, is the degree of \( p(X) \).) Similarly, geometrists could remove the exceptional behavior of “parallel lines” in the Euclidean plane once they completed it in a geometric way by adding “points at infinity,” arriving at the projective completion of the plane. One could then say that any two different lines intersect in exactly one point, and there was born a beautiful and symmetric area of mathematics, namely projective geometry.

For us, we may find a way out of our difficulties by using both kinds of completions. We first complete algebraically, using \( \mathbb{C} \) instead of \( \mathbb{R} \) (each set of polynomials \( p_1, \ldots, p_r \), with real or complex coefficients defines a variety \( V(p_1, \ldots, p_r) \) in \( \mathbb{C}^n \), and we also complete \( \mathbb{C} \) projectively to complex projective \( n \)-space, denoted \( \mathbb{P}^n(\mathbb{C}) \). The variety \( V(p_1, \ldots, p_r) \) in \( \mathbb{C}^n \) will be extended in \( \mathbb{P}^n(\mathbb{C}) \) by taking its topological closure. (We shall explain this further in a moment.) By extending our space and variety this way, we shall see that all exceptions to our “dimensional relation” will disappear, and algebraic varieties will behave just like subspaces of a vector space in this respect.

Hence, although in \( \mathbb{R}^2 \), \( X^2 + Y^2 - 1 \) defines a circle but \( X^2 + Y^2 + 1 \) the empty set, in our new setting each of these polynomials turns out to define a variety of (complex) codimension one in \( \mathbb{P}^1(\mathbb{C}) \), independent of what the “radius” of the circle might be. (The “complex dimension” of a variety \( V \) in \( \mathbb{C}^n \) is just one-half the dimension of \( V \) considered as a real point set; we shall see later that as a real point set, the dimension is always even. Also, even though the locus in \( \mathbb{C}^2 \) of \( X^2 + Y^2 = 1 \) does not turn out to look like a circle, we shall continue to use this term since the \( \mathbb{C}^2 \)-locus is defined by the same equation. Similarly, we shall use terms like curve or surface for complex varieties of complex dimension 1 and 2, respectively.)

In general, any nonconstant polynomial turns out to define a point set of complex codimension one in \( \mathbb{P}^n(\mathbb{C}) \), just as one (nontrivial) linear equation does in any vector space. A generalization of this vector space property is:
If $L_1$ and $L_2$ are subspaces of any $n$-dimensional vector space $k^n$ over $k$, then
\[
\text{cod}(L_1 \cap L_2) \leq \text{cod}(L_1) + \text{cod}(L_2)
\]
(cod $= \text{codimension}$).

For instance, any two 2-subspaces in $\mathbb{R}^3$ must intersect in at least a line. In $\mathbb{P}^n(\mathbb{C})$ this basic dimension relation holds even for arbitrary complex-algebraic varieties. Certainly nothing like this is true for varieties in $\mathbb{R}^2$. One can talk about disjoint circles in $\mathbb{R}^2$, or disjoint spheres in $\mathbb{R}^3$. These phrases make no sense in $\mathbb{P}^2(\mathbb{C})$ and $\mathbb{P}^3(\mathbb{C})$, respectively; the points missing in $\mathbb{R}^2$ or $\mathbb{R}^3$ simply are not seen because they are either “at infinity,” or have complex coordinates. (This will be made more precise soon.) Hence it turns out that what we see in $\mathbb{R}^n$ is just the tip of an iceberg—a rather unrepresentative slice of the variety at that—whose “true” life, from the algebraic geometer’s viewpoint, is lived in $\mathbb{P}^n(\mathbb{C})$.

# 2 The topology of a few specific plane curves

Suppose we have added the missing “points at infinity” to a complex algebraic variety in $\mathbb{C}^n$, thus getting a variety in $\mathbb{P}^n(\mathbb{C})$. It is natural to wonder what the entire “completed” curve looks like. We consider here only curves in $\mathbb{C}^2$ and in $\mathbb{P}^2(\mathbb{C})$; complex varieties of higher dimension have real dimension $\geq 4$ and our visual appreciation of them is necessarily limited. Even our complex curves live in real 4-space; our situation is somewhat analogous to an inhabitant of “Flatland” who lives in $\mathbb{R}^2$, when he attempts to visualize an ordinary sphere in $\mathbb{R}^3$. He can, however, see 2-dimensional slices of the sphere. Now in $X^2 + Y^2 + Z^2 = 1$, substituting a specific value $Z_0$ for $Z$ yields the part of the sphere in the plane $Z = Z_0$. Then if he lets $Z = T = \text{time}$, he can “visualize” the sphere by looking at a succession of parallel plane slices $X^2 + Y^2 = 1 - T^2$ as $T$ varies. He sees a “moving picture” of the sphere; it is a point when $T = -1$, growing to ever larger circles, reaching maximum diameter at $T = 0$, then diminishing to a point when $T = 1$.

Our situation is perhaps even more strictly analogous to his problem of visualizing something like a “warped circle” in 3-space (Figure 5). The

![Figure 5](image-url)
I: Examples of curves

Flatlander's moving picture of the circle's intersections with the planes $Z = \text{constant}$ will trace out a topological circle for him. He may not appreciate all the twisting and warping that the circle has in $\mathbb{R}^3$, but he can see its topological structure.

To get a topological look at our complex curves, let us apply this same idea to a hypersurface in complex 2-space. In $\mathbb{C}^2$, we will let the complex $X$-variable be $X = X_1 + iX_2$; similarly, $Y = Y_1 + iY_2$. We will let $X_2$ vary with time, and our "screen" will be real $(X_1, Y_1, Y_2)$-space. The intersection of the 3-dimensional hyperplane $X_2 = \text{constant}$ with the real 2-dimensional variety will in general be a real curve; we will then fit these curves together in our own 3-space to arrive at a 2-dimensional object we can visualize. As with the Flatlander, we will lose some of the warping and twisting in 4-space, but we will nonetheless get a faithful topological look, which we will be content with for now.

Since our complex curves will be taken in $\mathbb{P}^2(\mathbb{C})$, we first describe intuitively the little we need here in the way of projective completions. Our treatment is only topological here, and will be made fuller and more precise in Chapter II. We begin with the real case.

$\mathbb{P}^1(\mathbb{R})$: As a topological space, this is obtained by adjoining to the topological space $\mathbb{R}$ (with its usual topology) an "infinite" point, say $\infty$, together with a neighborhood system about $\infty$. For basic open neighborhoods we take

$$U_N(\infty) = \{\infty\} \cup \{r \in \mathbb{R} | |r| > N\} \quad N = 1, 2, 3, \ldots.$$  

We can visualize this more easily by shrinking $\mathbb{R}^1$ down to an open line segment, say by $x \to x/(1 + |x|)$. We may add the point at infinity by adjoining the two end points to the line segment and identifying these two points. In this way $\mathbb{P}^1(\mathbb{R})$ becomes, topologically, an ordinary circle.

$\mathbb{P}^2(\mathbb{R})$: First note that, except for $\mathbb{R}_X$, the 1-spaces $L_a = V(X + aY)$ of $\mathbb{R}^{XY}$ are parametrized by $a$; a different parametrization, $L_{a'} = V(a'X + Y)$, includes $\mathbb{R}_X$ (but not $\mathbb{R}_Y$). Then as a topological space, $\mathbb{P}^2(\mathbb{R})$ is obtained from $\mathbb{R}^2$ by adjoining to each 1-subspace of $\mathbb{R}^2$, a point together with a neighborhood system about each such point.

If, for instance, a given line is $L_{a_0}$, then for basic open neighborhoods about a given $P_{a_0}$ we take

$$U_N(P_{a_0}) = \bigcup_{|x| < 1/N} \{(P_x) \cup \{(x, y) \in L_x | |(x, y)| > N\}\} \quad N = 1, 2, 3, \ldots,$$

where $|(x, y)| = |x| + |y|$.

Similarly for lines parametrized by $a'$. (When $a$ and $a'$ both represent the same line $L_{a_0} = L_{a_0}$, the neighborhoods $U_N(P_{a_0})$ and $U_N(P_{a_0})$ generate the same set of open neighborhoods about $P_{a_0} = P_{a_0}$.)

Again, we can see this more intuitively by topologically shrinking $\mathbb{R}^2$ down to something small. For instance,

$$(x, y) \to \left(\frac{x}{1 + \sqrt{x^2 + y^2}}, \frac{y}{1 + \sqrt{x^2 + y^2}}\right)$$
maps $\mathbb{R}^2$ onto the unit open disk. Figure 6 shows this condensed plane together with some mutually parallel lines. (Two lines parallel in $\mathbb{R}^2$ will converge in the disk since distance becomes more "concentrated" as we approach its edge; the two points of convergence are opposite points. If, as in $\mathbb{P}^1(\mathbb{R})$, we identify these points, then any two "parallel" lines in the figure will intersect in that one point. Adding analogous points for every set of parallel lines in the plane means adding the whole boundary of the disk, with opposite (or antipodal) points identified. All these "points at infinity" form the "line at infinity," itself topologically a circle, hence a projective line $\mathbb{P}^1(\mathbb{R})$. Since this line at infinity intersects every other line in just one point, it is clear that any two different projective lines of $\mathbb{P}^2(\mathbb{R})$ meet in precisely one point.

$\mathbb{P}^1(\mathbb{C})$: Topologically, the "complex projective line" is obtained by adjoining to $\mathbb{C}$ an "infinite" point $P$; for basic open neighborhoods about $P$, take

$$U_X(P) = \{ P \} \cup \{ z \in \mathbb{C} | |z| > N \} \quad N = 1, 2, 3, \ldots.$$ 

Intuitively, shrink $\mathbb{C}$ down so it is an open disk, which topologically is also a sphere with one point missing (just as $\mathbb{R}$ is topologically a circle with one point missing). Adding this point yields a sphere.

$\mathbb{P}^2(\mathbb{C})$: As in the real case, except for the $X$-axis $\mathbb{C}_X$, the complex 1-spaces of $\mathbb{C}^2 = \mathbb{C}_X \oplus \mathbb{C}_Y$ are parametrized by $z$:

$$X + zY = 0 \quad \text{where} \quad z \in \mathbb{C};$$

another parametrization, $z'X + Y = 0$, includes $\mathbb{C}_X$ but not $\mathbb{C}_Y$. Then $\mathbb{P}^2(\mathbb{C})$ as a topological space is obtained from $\mathbb{C}^2$ by adjoining to each complex
1: Examples of curves

1-subspace \( L_0 = \mathcal{V}(X + 2Y) \) (or \( L_0 = \mathcal{V}(x'X + Y) \)) a point \( P_x \) (or \( P_z \)). A typical basic open neighborhood about a given \( P_{a_0} \) is

\[
U_N(P_{a_0}) = \bigcup_{|a-a_0|<1/N} \left\{ (P_x) \cup \{(z_1, z_2) \in L_0 | |(z_1, z_2)| > N \} \right\} \quad N = 1, 2, 3, \ldots
\]

where \(|(z_1, z_2)| = |z_1| + |z_2|\); similarly for neighborhoods about points \( P_{a_0} \).

Intuitively, to each complex 1-subspace and all its parallel translates, we are adding a single “point at infinity,” so that all these parallel lines intersect in one point. Each complex line is thus extended to its projective completion, \( \mathbb{P}^1(\mathbb{C}) \); and all points at infinity form also a \( \mathbb{P}^1(\mathbb{C}) \). As in \( \mathbb{P}^2(\mathbb{R}) \), any two different projective lines of \( \mathbb{P}^2(\mathbb{C}) \) meet in exactly one point.

The reader can easily verify from our definitions that each of \( \mathbb{R}, \mathbb{R}^2, \mathbb{C}, \mathbb{C}^2 \) is dense in its projective completion; hence the closure of \( \mathbb{C}^2 \) in \( \mathbb{P}^2(\mathbb{C}) \) is \( \mathbb{P}^2(\mathbb{C}) \), and so on. We shall likewise take the projective extension of a complex algebraic curve in \( \mathbb{C}^2 \) to be its topological closure in \( \mathbb{P}^2(\mathbb{C}) \).

We next consider some examples of projective curves using the slicing method outlined above.

**Example 2.1.** Consider the circle \( \mathcal{V}(X^2 + Y^2 - 1) \). Let \( X = X_1 + iX_2 \) and \( Y = Y_1 + iY_2 \). Then \( (X_1 + iX_2)^2 + (Y_1 + iY_2)^2 = 1 \). Expanding and equating real and imaginary parts gives

\[
X_1^2 - X_2^2 + Y_1^2 - Y_2^2 = 1, \quad X_1X_2 + Y_1Y_2 = 0. \tag{1}
\]

We let \( X_2 \) play the role of time; we start with \( X_2 = 0 \). The part of our complex circle in the 3-dimensional slice \( X_2 = 0 \) is then given by

\[
X_1^2 + Y_1^2 - Y_2^2 = 1, \quad Y_1Y_2 = 0. \tag{2}
\]

The first equation defines a hyperboloid of one sheet; the second one, the union of the \( (X_1, Y_1) \)-plane and the \( (X_1, Y_2) \)-plane (since \( Y_1 \cdot Y_2 = 0 \) implies \( Y_1 = 0 \) or \( Y_2 = 0 \)). The locus of the equations in (2) appears in Figure 7. It is

Figure 7
the union of the real circle $X_1^2 + Y_1^2 = 1$ (when $Y_2 = 0$) and the hyperbola $X_1^2 - Y_2^2 = 1$ (when $Y_1 = 0$). The circle is, of course, just the real part of the complex circle. The hyperbola has branches approaching two points at infinity, which we call $P_\infty$ and $P'_\infty$.

Now the completion in $\mathbb{P}_\mathbb{R}^2$ of the hyperbola is topologically an ordinary circle. Hence the total curve in our slice $X_2 = 0$ is topologically two circles touching at two points; this is drawn in Figure 8. The more lightly-drawn circle in Figure 8 corresponds to the (lightly-drawn) hyperbola in Figure 7.

Now let’s look at the situation when “time” $X_2$ changes a little, say to $X_2 = \varepsilon > 0$. This defines the corresponding curve

$$X_1^2 + Y_1^2 - Y_2^2 = 1 + \varepsilon^2, \quad \varepsilon X_1 + Y_1 Y_2 = 0.$$

The first surface is still a hyperboloid of one sheet; the second one, for $\varepsilon$ small, in a sense “looks like” the original two planes. The intersection of these two surfaces is sketched in Figure 9. The circle and hyperbola have split into two disjoint curves. We may now sketch these disjoint curves in on Figure 8; they always stay close to the circle and hyperbola. If we fill in all
such curves corresponding to $X_2 = \text{constant}$, we will fill in the surface of a sphere. The curves for nonnegative $X_2$ are indicated in Figure 10.

For $X_2 < 0$, one gets curves lying on the other two quarters of the sphere. We thus see (and will rigorously prove in Section II.10) that all these curves fill out a sphere. We thus have the remarkable fact that the complex circle $V(X^2 + Y^2 - 1)$ in $\mathbb{P}^2(\mathbb{C})$ is topologically a sphere.

From the complex viewpoint, the complex circle still has codimension 1 in its surrounding space.

**Example 2.2.** Now let us look at a circle of "radius 0," $V(X^2 + Y^2)$. The equations corresponding to (1) are

$$X_1^2 - X_2^2 + Y_1^2 - Y_2^2 = 0, \quad X_1X_2 + Y_1Y_2 = 0.$$  \hspace{1cm} (3)

The part of this variety lying in the 3-dimensional slice $X_2 = 0$ is then given by

$$X_1^2 + Y_1^2 - Y_2^2 = 0, \quad Y_1Y_2 = 0.$$  \hspace{1cm} (4)

The first equation defines a cone; the second one defines the union of two planes as before. The simultaneous solution is the intersection of the cone and planes. This consists of two lines (See Figure 11). The projective closure of each line is a topological circle, so the closure of the two lines in this figure consists of two circles touching at one point. This can be thought of as the limit figure of Figure 8 as the horizontal circle's radius approaches zero.

When $X_2 = \varepsilon$, the saddle-surface defined by $\varepsilon X_1 + Y_1Y_2 = 0$ intersects the one-sheeted hyperboloid given by $X_1^2 + Y_1^2 - Y_2^2 = \varepsilon^2$. As before, their intersection consists of two disjoint real curves, which turn out to be lines (Figure 12); just as in the first example, as $X_2$ varies, the curves fill out a 2-dimensional topological space which is like Figure 10, except that the radius of the horizontal circle is 0 (Figure 13). To keep the figure simple, only curves for $X_2 \geq 0$ have been sketched; they cover the top half of the upper
Examples of curves

sphere and the bottom half of the lower sphere, the other parts being covered when \( X_2 < 0 \). Hence: The complex circle of "zero radius" \( V(X^2 + Y^2) \) in \( \mathbb{P}^2(\mathbb{C}) \) is topologically two spheres touching at one point.

In the complex setting, we see that instead of the dimension changing as soon as the "radius" becomes zero, the complex circle remains of codimension 1, so that one equation \( X^2 + Y^2 = 0 \) still cuts down the (complex) dimension by one.

Incidentally, here is another fact that one might notice: In Example 2.1, \( V(X^2 + Y^2 - 1) \), the sphere is in a certain intuitive sense "indecomposable," while in Example 2.2, the figure is in a sense "decomposable," consisting of two spheres which touch at only one point. But look at the polynomial \( X^2 + Y^2 - 1 \); it is "indecomposable" or irreducible in \( \mathbb{C}[X, Y] \). And the polynomial \( X^2 + Y^2 \) is "decomposable," or reducible, \( X^2 + Y^2 = (X + iY)(X - iY) \) in fact, \( X^2 + Y^2 + \gamma \) is always irreducible in \( \mathbb{C}[X, Y] \) if \( \gamma \neq 0 \). (A proof may be given similar in general spirit to that in Footnote 1.) Hence we should suspect that any complex circle with "nonzero radius" should be somehow irreducible. We shall see later that in an appropriate sense this is indeed true. By the way, \( X^2 + Y^2 = (X + iY)(X - iY) \) expresses that \( V(X^2 + Y^2) \) is just the union \( V(X + iY) \cup V(X - iY) \). Each of these last varieties is a projective line, which is topologically a sphere; and any two projective lines touch in exactly one point in \( \mathbb{P}^2(\mathbb{C}) \). This is a very different way of arriving at the topological structure of \( V(X^2 + Y^2) \).

Example 2.3. Let us look next at a circle of "pure imaginary radius," \( V(X^2 + Y^2 + 1) \). Separating real and imaginary parts gives

\[
X_1^2 - X_2^2 + Y_1^2 - Y_2^2 = -1, \quad X_1X_2 + Y_1Y_2 = 0. \tag{6}
\]

At \( X_2 = 0 \) this defines the part common to a hyperboloid of two sheets and the union of two planes. This is a hyperbola. Its two branches start approaching each other as \( X_2 \) increases, finally meeting at \( X_2 = 1 \) (the hyperboloid of two sheets has become the cone \( X_1^2 + Y_1^2 - Y_2^2 = 0 \)). Then for \( X_2 > 1 \), we are back to the same kind of behavior as for \( V(X^2 + Y^2 - 1) \) when \( X_2 > 0 \). Figure 14, analogous to Figures 10 and 13, shows how we end up with a sphere. Later we will supplement this result by proving:

Topologically, \( V(X^2 + Y^2 + \gamma) \) in \( \mathbb{P}^2(\mathbb{C}) \) is a sphere iff \( \gamma \neq 0 \).

\footnote{If \( X^2 + Y^2 - 1 \) were factorable into terms of lower degree, it would have to be of the form

\[
X^2 + Y^2 - 1 = (aX + bY + c)(X + \frac{1}{b}Y - \frac{1}{c}) \quad a, b, c \neq 0; \tag{5}
\]

this follows from multiplying and equating coefficients. Also, equating \( X \)-terms yields \( 0 = -(a/c) + (c/a) \), or \( c^2 = a^2 \). Similarly, \( c^2 = b^2 \), so \( a^2 = b^2 \), which in turn yields a term \( \pm 2XY \) on the right-hand side of (5), a contradiction.}
Do other familiar topological spaces arise from looking at curves in $\mathbb{P}^2(\mathbb{C})$? For instance, is a torus (a sphere with one "handle"—that is, the surface of a doughnut) ever the underlying topological space of a complex curve? More generally, how about a sphere with $g$ handles in it (topological manifold of genus $g$)? Let us consider the following example:

**Example 2.4.** The real part of the curve $\mathcal{V}(Y^2 - X(X^2 - 1))$, frequently encountered in analytic geometry, appears in Figure 3. (The reader will learn, at long last, what happens in those mysterious "excluded regions" $-\infty < X < -1$ and $0 < X < 1$.)

Separating real and imaginary parts in $Y^2 - X(X^2 - 1) = 0$ gives

\[
Y_1^2 - Y_2^2 = X_1^3 - 3X_1^2X_2 - X_1,
\]
\[
2Y_1Y_2 = 3X_1^2X_2 - X_2^3 - X_2. \tag{7}
\]

When $X_2 = 0$, this becomes

\[
Y_1^2 - Y_2^2 = X_1^3 - X_1, \quad Y_1Y_2 = 0.
\]

Then either $Y_1 = 0$ or $Y_2 = 0$. When $Y_2 = 0$, the other equation becomes $Y_1^2 = X_1^3 - X_1$. The sketch of this is of course again in Figure 3—that is, when $X_2 = Y_2 = 0$ we get the real part of our curve. When $Y_1 = 0$, we get a "mirror image" of this in the $(X_1, Y_2)$-plane. The total curve in the slice $X_2 = 0$ appears in Figure 15.

Note that in the right-hand branch, $Y_1$ increases faster than $X_1$ for $X_1$ large, so the branch approaches the $Y_1$-axis. Similarly, the left-hand branch approaches the $Y_2$-axis. But in $\mathbb{P}^2(\mathbb{C})$, exactly one infinite point is added to each complex 1-space, and the $(Y_1, Y_2)$-plane is the 1-space $Y = 0$. Hence the
two branches meet at a common point $P_\infty$. We may topologically redraw our curve in the 3-dimensional slice as in Figure 16.

By letting $X_2 = \epsilon$ in (7) and using continuity arguments, one sees that the curves in the other 3-dimensional slices fill in a torus. In Figure 17, solid lines on top and dotted lines on bottom come from curves for $X_2 \geq 0$. The rest of the torus is filled in when $X_2 < 0$. The real part of the graph of $Y^2 = X(X^2 - 1)$ is indeed a small part of the total picture!

We now generalize this example to show we can get as underlying topological space, a “sphere with any finite number of handles”; this is the most general example of a compact connected orientable 2-dimensional manifold. Such a manifold is completely determined by its genus $g$. (We take this up later on; Figure 19 shows such a manifold with $g = 5$.)
Example 2.5. \( \mathcal{V}(Y^2 - X(X^2 - 1) \cdot (X^2 - 4) \cdot \ldots \cdot (X^2 - g^2)) \). For purposes of illustration we use \( g = 5 \). The sketch of the corresponding real curve appears in the \((X_1, Y_1)\)-plane of Figure 18. The whole of Figure 18 represents the curve in the slice \( X_2 = 0 \).

Note the analogy with Figure 15. As before, the branches in Figure 18 meet at the same point at infinity. This may be topologically redrawn as in Figure 19, where also the curves for \( X_2 \geq 0 \) have been sketched in.

We now see that looking at "loci of polynomials" from the complex viewpoint automatically leads us to topological manifolds! Incidentally, these last manifolds of arbitrary genus are intuitively "indecomposable" in a way that the sphere was earlier, so we have good reason to suspect that any polynomial \( Y^2 - X(X^2 - 1) \cdot (X^2 - 4) \cdot \ldots \cdot (X^2 - g^2) \) is irreducible in
$\mathbb{C}[X, Y]$. This is in fact so. Note, however, that a polynomial having as repeated factors an irreducible polynomial may still define an indecomposable object. (For example, $V(X - Y) = V((X - Y)^3)$ is topologically a sphere in $\mathbb{P}^2(\mathbb{C})$.) We also recall that if we take a finite number of irreducible polynomials and multiply them together, the irreducibles' identities are not obliterated, for we can refactor the polynomial to recapture the original irreducibles (by "uniqueness"). The same behavior holds at the geometric level; each topological object in $\mathbb{P}^2(\mathbb{C})$ coming from a (nonconstant) polynomial $p \in \mathbb{C}(X, Y)$ is 2-dimensional, but it turns out that objects coming from different irreducible factors of $p$ touch in only a finite number of points, and that removing these points leaves us with a finite number of connected, disjoint parts. These parts are in 1:1 correspondence with the distinct irreducible factors of $p$. For instance, $V(p)$, with

$$p = (Y^2 - X(X^2 - 1)(X^2 - 4)) \cdot Y \cdot (Y - 1),$$
turns out to look topologically like Figure 20; it falls into three parts, the
two spheres corresponding to the factors $Y$ and $(Y - 1)$, and the manifold
of genus 2, corresponding to the $5$th degree factor. The spheres touch each
other in one point, and each sphere touches the third part in 5 points.

**Example 2.6.** We cannot leave this section of examples without at least
briefly mentioning curves with singularities; an example is given by the
alpha curve $V(Y^2 - X^2(X + 1))$ (Figure 2). Separating real and imaginary
parts of $Y^2 - X^2(X + 1) = 0$ and setting $X_2 = 0$ gives us a curve
sketched in Figure 21. The two branches again meet at one point at infinity,
$P_\infty$, and the other curves $X_2 = \text{constant}$ fit together as in Figure 22. Topo-
logically this is obtained by taking a sphere and identifying two points.
Note that $Y^2 - X^2(X + 1)$ is just the limit of $Y^2 - X(X - \varepsilon)(X + 1)$ as
$\varepsilon \to 0$. One can think of Figure 22 as being the result of taking the topological
circle in Figure 17 between the roots 0 and 1 and “squeezing this circle
to a point.” Also note that this “squeezing” process not only introduces a
singularity, but has the effect of decreasing the genus by one; the genus of
$V(Y^2 - X(X^2 - 1))$ is 1, while $V(Y^2 - X^2(X + 1))$ is a sphere (genus 0) with
two points identified. One may instead choose to squeeze to a point, say, the
circle in Figure 17 between roots $-1$ and 0; this corresponds to
$V(Y^2 - X^2(X - 1))$. Its sketch in real $(X_1, Y_1)$-space is just the “mirror
image” of Figure 2. Squeezing this middle circle to a point gives a sphere
with the north and south poles identified to a point; the reader may wish
to check that these two different ways of identifying two points on a sphere
yield homeomorphic objects.

What if one brings together *all* three zeros of $X(X + 1)(X - 1)$? That is,
what does $V(\lim_{\varepsilon \to 0}[Y^2 - X(X + \varepsilon)(X - \varepsilon)]) = V(Y^2 - X^3)$ look like?
Of course its real part is just the cusp of Figure 1; the origin is again an
example of a singular point. As it turns out, $V(Y^2 - X^3)$ is topologically a
sphere (Exercise 2.2).

![Figure 21](image-url)
Examples of curves

After seeing all these examples, the reader may well wonder:

What is the most general topological object in \( \mathbb{P}^2(\mathbb{C}) \) defined by a (nonconstant) polynomial \( p \in \mathbb{C}[X, Y] \)?

The answer is:

**Theorem 2.7.** If \( p \in \mathbb{C}[X, Y] \setminus \mathbb{C} \) is irreducible, then topologically \( V(p) \) is obtained by taking a real 2-dimensional compact, connected, orientable manifold (this turns out to be a sphere with \( g < \infty \) handles) and identifying finitely many points to finitely many points: for any \( p \in \mathbb{C}[X, Y] \setminus \mathbb{C} \), \( V(p) \) is a finite union of such objects, each one furthermore touching every other one in finitely many points.

We remark that a (real, topological) \( n \)-manifold is a Hausdorff space \( M \) such that each point of \( M \) has an open neighborhood homeomorphic to an open ball in \( \mathbb{R}^n \). For definitions of connectedness and orientability, see Definitions 8.1 and 9.3, and Remark 9.4 of Chapter II.

One of the main aims of Chapter II is to prove this theorem.

Example 2.8. In Figure 23 a real 2-dimensional compact, connected, orientable manifold of genus 4 has had 7 points identified to 3 points (3 to 1, 2 to 1, and 2 to 1).

Remark 2.9. We do not imply that every topological object described above actually is the underlying space of some algebraic curve in \( \mathbb{P}^2(\mathbb{C}) \). However, one can, by identifying roots of \( Y^2 - X(X^2 - 1)\ldots(X^2 - g^2) \), manufacture spaces having any genus, with any number of distinct “2 to 1”
identifications. But how about any number of "3 to 1", or "4 to 1" identifications, etc.? And in just how many points can we make one such "indecomposable" space touch another? Even partial answers to such questions involve a careful study of such things as Bézout's theorem, Plücker's formulas, and the like.

**EXERCISES**

2.1 Show, using the "slicing method" of this section, that the completion in \( \mathbb{P}^2(\mathbb{C}) \) of the complex parabola \( \mathcal{V}(Y - X^2) \) and the complex hyperbola \( \mathcal{V}(X^2 - Y^2 - 1) \) are topologically both spheres.

2.2 Draw figures corresponding to Figures 7-10 to show that the completion in \( \mathbb{P}^2(\mathbb{C}) \) of \( \mathcal{V}(Y^2 - X^3) \) is a topological sphere. Compare your figures with those for \( \mathcal{V}(Y^2 - X^2(X + c)) \), as \( c \rightarrow 0 \) approaches zero.

2.3 Establish the topological nature of the completion in \( \mathbb{P}^2(\mathbb{C}) \) of \( \mathcal{V}(X^2 - Y^2 + r) \), as \( r \) takes on real values in \([-1, 1]\).

3 Intersecting curves

The fact that any two "indecomposable" algebraic curves in \( \mathbb{P}^2(\mathbb{C}) \) must intersect (as implied by the description in the last section), follows at once from the dimension relation

\[
\text{cod}(\mathcal{V}(p_i) \cap \mathcal{V}(p_j)) \leq \text{cod} \mathcal{V}(p_i) + \text{cod} \mathcal{V}(p_j),
\]

which means, in our case, \( \text{cod}(\mathcal{V}(p_i) \cap \mathcal{V}(p_j)) \leq 2 \), or

\[
\text{dim}(\mathcal{V}(p_i) \cap \mathcal{V}(p_j)) \geq 0. \text{ Hence } \mathcal{V}(p_i) \cap \mathcal{V}(p_j) \neq \emptyset \quad (\text{dim } \emptyset = -1).
\]

**Example 3.1.** For two parallel complex lines in \( \mathbb{C}^2 \), the above amounts to a restatement that these lines must intersect in \( \mathbb{P}^2(\mathbb{C}) \).
I: Examples of curves

Example 3.2. Any complex line and any complex circle in \( \mathbb{P}^2(\mathbb{C}) \) must intersect. One can actually see, in Figure 7, how any parallel translate of the complex \( Y \)-plane along the \( X_1 \)-axis still intersects the circle \( V(X^2 + Y^2 - 1) \), either in the \( (X_1, Y_1) \)-plane (as we usually see the intersection), or in the hyperbola in the \( (X_1, Y_2) \)-plane, for \( |X_1| > 1 \).

Example 3.3. Another case may be of some interest. Let us consider one curve which is a complex line, say \( V(Y) \). Let another curve be \( V(Y - q(X)) \), where \( q \) is a polynomial in \( X \) alone. Then the graph of \( Y = q(X) \) in \( \mathbb{C}_{XY} \) is homeomorphic to \( \mathbb{C}_X \); one can then easily check that \( V(Y - q(X)) \) is a topological sphere in \( \mathbb{P}^2(\mathbb{C}) \), as is \( V(Y) \). By our dimension relation, these spheres must intersect, perhaps as in Figure 24.

![Figure 24](image)

Does this result sound familiar? It is very much like the fundamental theorem of algebra (every nonconstant \( q(X) \in \mathbb{C}[X] \) has a zero in \( \mathbb{C} \)). This famous result can now be put into the \( \mathbb{P}^2(\mathbb{C}) \) setting. If we do this, then in stating the fundamental theorem of algebra,

(a) There is no need to assume \( q(x) \in \mathbb{C}[X] \) is nonconstant; if it is constant, we will have two lines which either coincide or which intersect at one point at infinity.

(b) There is no need to assume the given line is \( V(Y)(= \mathbb{C}_X) \). Any line, in any position, will do as well.

(c) There is no need to assume one polynomial is linear (thus describing a line), or that the other one must be of such a very special form as \( Y - q(X) \) (describing the graph of a function: \( \mathbb{C} \to \mathbb{C} \)). The loci of any two curves in \( \mathbb{P}^2(\mathbb{C}) \) must intersect.

Of course, the reader might argue that our dimension relation fails to give us certain other information which the fundamental theorem of algebra readily provides. For instance, the fundamental theorem can be stated more
informatively as: “Any nonzero polynomial \( q(X) \in \mathbb{C}[X] \) has exactly \( \deg q(X) \) zeros when counted with multiplicity.” However, our dimension relation can be extended in an analogous way.

To see how, first consider \( q(X) \) again. We may look at the multiplicity \( r_i \) of the zero \( c_i \) in \( q(X) = (X - c_1)^{r_1} \cdots (X - c_k)^{r_k} \) in the following geometric way. Let \( a \in \mathbb{C} \setminus \{0\} \), and let \( L_a \) be the complex line \( Y = a \) in \( \mathbb{C} \times \mathbb{C} \). Then the following holds for all \( a \) sufficiently small:

Of those points in which \( L_a \) intersects the graph of \( Y = q(X) \), there are exactly \( r_i \) of them clustered close to the point \( (c_i, 0) \), where \( r_i \) is the multiplicity of the zero \( c_i \) in \( q(X) \).

(A proof of a more general version of this fact will be provided in Section IV.6.)

**Example 3.4.** The point \( 0 \in \mathbb{C} \) is a double root of the polynomial equation \( Y = q(X) = X^2 \). For \( a \neq 0 \), \( L_a \) intersects the parabola in two distinct points. (They have complex \( X \)-coordinates if \( a \neq \mathbb{R}^+ \).) See Figure 25.

![Figure 25](image)

For small \( a \), these two points cluster close to \( (0, 0) \); in this way our zero of multiplicity two can be looked at as the limit of two single points which have coalesced. The fundamental theorem says that the sum of all the multiplicities \( r_i \) at all zeros \( c_1, \ldots, c_k \) of the polynomial \( q(X) \) is \( \deg q(X) \). The situation of two curves \( C_1, C_2 \) in \( \mathbb{C}^2 \) is very much the same; if \( P \) is an isolated point of intersection of \( C_1 \) and \( C_2 \), there will be a certain integer \( n_P \) so that the following holds:

**Example 3.5** For most sufficiently small translates of \( C_1 \) or \( C_2 \), of those points in which the translated curves intersect, there will be exactly \( n_P \) distinct points clustered close to \( P \).
1: Examples of curves

The meaning of “most” above will be made precise in Chapter IV. The reader can get the basic idea via some examples. The integer $n_p$ is called the intersection multiplicity of $C_1$ and $C_2$ at $P$.

**Example 3.6.** Consider the intersection at $(0, 0) \in \mathbb{C}^2$ of the two alpha curves $C_1 = \mathbb{V}(Y^2 - X^2(X + 1))$ and $C_2 = \mathbb{V}(X^2 - Y^2(Y + 1))$ (Figure 26). In Figure 27 $C_2$ has been translated upward a little; there are four distinct points clustered about $(0, 0)$. One can see that “most” translates will yield four points this way. In certain special directions one can get fewer than four points (but never more, nearby). Figure 28 gives an example of this. But in a certain sense the topmost of these clustered points is still “multiple” — that is, a little push up or down of either curve will further separate that point so we again end up with a total of four points.
Of course, in most cases, the intersection points have complex coordinates; the above example is quite special in the respect that all intersection points are real. Also, we have not shown by our real pictures that there cannot be more than four points clustered near (0, 0); for this, we need to determine from the equations of the curves all possible simultaneous solutions near (0, 0). But we can translate this geometric idea of “perturbing one curve slightly to separate the points” into algebraic terms; this is done in Chapter IV when we take up intersection multiplicity formally.

An extension of the dimension relation for curves which corresponds to our geometric form of the fundamental theorem of algebra is then the following:

Let \( p_1, \ldots, p_k \) be all the points of intersection of two curves \( V(p_1) \) and \( V(p_2) \) in \( \mathbb{P}^2(\mathbb{C}) \), where \( p_1 \) and \( p_2 \) have no repeated factors, and no factor in common. Then the total number of points of intersection of \( V(p_1) \) and \( V(p_2) \), counted with multiplicity, is \( (\deg p_1) \cdot (\deg p_2) \). (Often the number of points counted with multiplicity is called the degree of the intersection and one writes

\[
\deg(V(p_1) \cap V(p_2)) = (\deg p_1) \cdot (\deg p_2).
\]

This elegant and central result is known as Bézout’s theorem, after its discoverer, the French mathematician E. Bézout (1730–1783).

**Remark 3.7.** We must assume \( p_i \) is a polynomial of lowest degree defining \( V(p_i) \), i.e., that \( p_i \) does not have repeated factors. For instance the \( X \) and \( Y \) axes, which are \( V(Y) \) and \( V(X^2) \), intersect in just one point instead of \( 1 \cdot 2 = 2 \) points. However, using \( V(X) \) yields \( 1 \cdot 1 = 1 \) point. The assumption that \( p_1 \) and \( p_2 \) have no factor in common is of course needed to make \( V(p_1) \cap V(p_2) \) finite.

**Example 3.8.** Assuming (3.5), we can now see more precisely the relation between Bézout’s theorem and the fundamental theorem of algebra. Assume, in \( Y = p(X) \), that \( p \) is not constant. Then \( Y \) eventually increases like \( X \) to a positive power; hence the graph \( V(Y - p(X)) \) cannot intersect the \( X \)-axis at infinity—that is, all intersections take place in \( \mathbb{C}_{XY} \). Secondly, since \( p \) is nonconstant, \( \deg(Y - p(X)) = \deg p(X) \). Finally the \( X \)-axis, i.e., \( Y = 0 \), has degree 1. Hence the number of intersections of the graph in \( \mathbb{C}_{XY} \) with \( C_X \) is \( 1 \cdot \deg p(X) = \deg p(X) \).

**Example 3.9.** Consider two ellipses as in Figure 29. Each ellipse is defined by a polynomial of degree two, and the total number of intersection points is \( 2 \cdot 2 = 4 \). As the horizontal ellipse is translated upward, we get first one double point at top, and two single points; then two complex points and two real ones: then 2 complex and 1 real double; and finally, as the ellipses separate entirely in the real plane, we have four complex points of intersection.
1. Examples of curves

\[ \text{Figure 29} \]

**Example 3.10.** Consider the curves $V(Y^2 - 5X^2(X + 1))$ and $V(Y^2 - X - 1)$. Their degrees are 3 and 2, so there should be a total of six intersection points. In Figure 30 four single intersections and one double intersection appear.

\[ \text{Figure 30} \]

**Exercises**

3.1 Suppose that curves $C_1$ and $C_2$ in $\mathbb{P}^2(\mathbb{C})$ have exactly $p$ points of intersection (counting multiplicity), where $p$ is a prime. Show that either $C_1$ or $C_2$ must be a topological sphere. (Assume Bézout’s theorem.)

3.2 Consider the curves $V(Y^2 - X^3)$ and $V(Y^2 - 2X^3)$ in $\mathbb{C}_X$. Find all points of intersection in $\mathbb{C}_X$ near $(0, 0)$, after one curve is given an arbitrarily small nonzero translation. How many points of intersection (counted with multiplicity) are there at the “line at infinity” of $\mathbb{C}_X$?

3.3 Consider the complex circles $V(X^2 + Y^2 - 1)$ and $V(X^2 + Y^2 - 4)$ in $\mathbb{P}^2(\mathbb{C})$. Where are their points of intersection? Are all four points of intersection distinct?
4 Curves over \( \mathbb{Q} \)

Perhaps enough has been said about plane curves to give the reader a first bit of intuition about them. Let us now look in perspective at what we've done so far. We started out using \( \mathbb{R} \) as our groundfield, and were led to \( \mathbb{C} \) as a ground field where varieties were better able to express an important side of their nature. We then saw topologically, just what certain complex curves look like, and we got a look at how they behave under intersection. This immediately suggests many more questions: What do varieties of \textit{arbitrary} dimension in \( \mathbb{P}^n(\mathbb{C}) \) look like? How do they behave under intersection? Under union? Can one "multiply" varieties as one does topological spaces to get product varieties? Are there natural maps from one variety to another, as there are continuous maps from one topological space to another? How do they behave? Can one put more structure on an algebraic variety (as one does with topological spaces) to arrive at "algebraic groups," etc.? An important part of algebraic geometry consists in exploring such questions. Also, if going from ground field \( \mathbb{R} \) to \( \mathbb{C} \) allowed varieties a fuller expression of a certain aspect of their nature, might possibly using other ground fields allow varieties to express a different side of their nature? This is indeed so, and no introductory tour of algebraic varieties would be complete without at least mentioning varieties over ground fields other than \( \mathbb{R} \) and \( \mathbb{C} \).

For purposes of illustration, let us consider the field \( \mathbb{Q} \) of rational numbers. Even asking for the appearance of a few specific curves in \( \mathbb{Q}^2 \) will show us a totally new aspect of algebraic varieties. For many curves defined by polynomials in \( \mathbb{Q}[X, Y] \), their appearance is the "same" as in \( \mathbb{R}^2 \)—that is, if \( C \) is the locus in \( \mathbb{R}^2 \) of \( p(X, Y) \in \mathbb{Q}[X, Y] \), then \( C \cap \mathbb{Q}^2 \) is dense in \( C \). But for other curves the appearances of \( C \) in \( \mathbb{Q}^2 \) versus \( \mathbb{R}^2 \) are completely different.

\textbf{Example 4.1.} \( V = \mathcal{V}(Y - X^2) \subset \mathbb{Q}^2 \); with the usual topology of \( \mathbb{R}^2 \), this \( V \) is dense in the variety \( V' = \mathcal{V}(Y - X^2) \subset \mathbb{R}^2 \), so \( V \) "looks like" the parabola \( V' \) in \( \mathbb{R}^2 \). The density of \( V \) is easily verified because \( y = x^2 \) is a continuous function, \( \mathbb{Q} \) is dense in \( \mathbb{R} \), and \( x \in \mathbb{Q} \) implies \( (x, x^2) \in V \subset \mathbb{Q}^2 \).

\textbf{Example 4.2.} In a similar way, one sees that if \( p(X) \in \mathbb{Q}[X] \), then the "graph-variety" \( \mathcal{V}(Y - p(X)) \) in \( \mathbb{Q}^2 \) is dense in the corresponding graph in \( \mathbb{R}^2 \).

\textbf{Example 4.3.} \( V = \mathcal{V}(Y^2 - X^3) \subset \mathbb{Q}^2 \) is dense in \( V' = \mathcal{V}(Y^2 - X^3) \subset \mathbb{R}^2 \); so \( V \) also looks like the cusp \( V' \) in \( \mathbb{R}^2 \). This is true because the squares of rational numbers form a dense subset of \( \mathbb{R}^+ \); if \( s \) is a square of a rational, then \( (s, \pm s^{3/2}) \in V \subset \mathbb{Q}^2 \).

\textbf{Example 4.4.} The "rational circle" \( V = \mathcal{V}(X^2 + Y^2 - 1) \subset \mathbb{Q}^2 \) also turns out to be dense in the corresponding real circle of \( \mathbb{R}^2 \), but this time the
I: Examples of curves

reasoning is more subtle. Let \((r, s) \in V(X^2 + Y^2 - 1) \subset \mathbb{Q}^2\). We may assume without loss of generality that \(r\) and \(s\) have the same denominator, \(r = a/c, s = b/c, a, b, c,\) integers. Then \(r^2 + s^2 = 1\) implies \(a^2 + b^2 = c^2\), i.e., that \((a, b, c)\) is a Pythagorean triple, meaning that \(|a|, |b|, |c|\) form the lengths of the sides of a right triangle (a Pythagorean triangle). Now the number-theoretic problem of finding all Pythagorean triangles was solved already by Euclid’s time. The solution says, in essence:

All Pythagorean triples \((a, b, c)\) are obtained from

\[
\begin{align*}
\frac{a}{c} &= \frac{v^2 - u^2}{u^2 + v^2}, & \frac{b}{c} &= \frac{2uv}{u^2 + v^2},
\end{align*}
\]

where \(u, v\) range through all integers \((u, v \neq 0)\).

The question of whether the points \((r, s) = (a/c, b/c)\) are dense in the real circle is evidently the same as:

Can the slope

\[
\frac{a/c}{b/c} = \frac{a}{b} = \frac{v^2 - u^2}{2uv}
\]

be made arbitrarily close to any preassigned slope \(m \in \mathbb{R}\)?

But

\[
\frac{v^2 - u^2}{2uv} \approx m \quad \text{implies} \quad \frac{v}{u} - \frac{u}{v} \approx 2m,
\]

meaning that for some rational \(x = u/v, \ x - (1/x) \approx 2m\). This implies \(x^2 - 2mx - 1 \approx 0\) which further implies \(x \approx m \pm \sqrt{m^2 + 1}\). But surely any \(m \pm \sqrt{m^2 + 1} \in \mathbb{R}\) can be approximated to any degree of accuracy by a rational number.

Geometrically, the density of the rational circle in the ordinary real one says that there are Pythagorean triangles arbitrarily close in shape to any given right triangle.

EXAMPLE 4.5. The last example involved a solution to an honest problem in number theory. Finding out exactly what the rational curve \(V(X^n + Y^n - 1) \subset \mathbb{Q}^2\) looks like for all integers \(n > 2\) is probably the most famous unsolved problem in all mathematics; it is equivalent to Fermat’s last theorem. This conjecture says that in \(\mathbb{Q}, \) for any \(n \geq 2, V(X^n + Y^n - 1)\) consists of just the four points \(\{(1, 0)\} \cup \{(0, 1)\} \cup \{(-1, 0)\} \cup \{(0, -1)\}\) if \(n\) is even, and consists of just the two points \(\{(1, 0)\} \cup \{(0, 1)\}\) if \(n\) is odd. This is vastly different from the corresponding real curves (Figures 31 and 32). It looks more and more square-shaped as \(n\) becomes large and even (Figure 31), and looks like the sketch in Figure 32 for \(n\) large and odd.
From just these few examples, the reader has perhaps already guessed that by using \( \mathbb{Q} \) as ground field, varieties express a strong number-theoretic aspect. (This is true also of algebraic extensions of \( \mathbb{Q} \) and finite fields). In fact, much of the modern work in number theory, including looking anew at some of the old classical problems, has been done by looking at number theory geometrically from the vantage point of algebraic geometry.

In this chapter we have tried to give the reader a little feeling for algebraic varieties, mostly by example. A large part of algebraic geometry consists in rigorizing and extending these ideas, not only to arbitrary complex varieties in \( \mathbb{P}^n(\mathbb{C}) \) or to arbitrary varieties in an analogous \( \mathbb{P}^n(k) \), but also to finding appropriate analogues when \( k \) is replaced by quite general commutative rings with identity. From our examples of switching from \( \mathbb{R} \) to \( \mathbb{C} \) or to \( \mathbb{Q} \), the reader may be somewhat convinced that such generalization is not just for generalization's sake, but frequently leads to important connections with other areas of mathematics.

**EXERCISES**

4.1 Is the set \( \mathcal{V}(X^2 - Y^2 - 1) \subset \mathbb{Q}^2 \) dense in \( \mathcal{V}(X^2 - Y^2 - 1) \subset \mathbb{R}^2 \)? What about the curves \( \mathcal{V}(X^n - Y^n - 1) \), for \( n > 2 \)?

4.2 Is the alpha curve \( \mathcal{V}(Y^2 - X^2(X + 1)) \subset \mathbb{Q}^2 \) dense in the corresponding curve in \( \mathbb{R}^2 \)?

4.3 Find an ellipse \( \mathcal{V}(aX^2 + bY^2 - 1) \) \( (a, b \in \mathbb{Q} \setminus \{0\}) \) whose graph in \( \mathbb{Q}^2 \) is the empty set. [Hint: Any solution \( X_i = x_i \) in \( \mathbb{Z} \) of \( n_1X_1^2 + n_2X_2^2 + n_3X_3^2 = 0 \) \( (n_i \in \mathbb{Z}) \) implies, for every integer \( n \), a solution in \( \mathbb{Z}/(n) \) of \( n_1X_1^2 + n_2X_2^2 + n_3X_3^2 \equiv 0 \) \( \pmod{n} \).] Can we assume \( a = b \)?