The Risk of Hitting the Zero Lower Bound and the Optimal Inflation Target*

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Abstract

I examine the optimal inflation target in a dynamic stochastic New Keynesian model featuring an occasionally binding zero lower bound on nominal interest rate (ZLB). To this end, I first calibrate the shock to generate the risk of hitting the ZLB that matches the U.S. data, based on a fully nonlinear method. I then resolve the model with different inflation targets and find that the optimal target is 3.4%. In addition, the optimal inflation target is a nonlinear function of the risk of hitting the ZLB and inflation indexation. It is always greater than 2% if the risk is greater than 2.5% or if the inflation indexation is larger than 0.5. Finally, the linear-quadratic (LQ) approach overestimates the true optimal inflation target. In particular, based on the benchmark calibration, it generates an optimal target of 5.5%, compared to 3.4% found by the fully-nonlinear method.

JEL classification: E52, E58.

Keywords: Zero lower bound; Optimal inflation target; Inflation indexation; Rotemberg price adjustments.

1 Introduction

Inflation targeting is a very important monetary policy strategy that helps create a nominal anchor to tie down the price level, so that central bankers are able to obtain their price stability objectives. Since the early 1990s, advanced economies started using this strategy, either explicitly, i.e. New Zealand, Canada and the UK, or implicitly, for example the US. Although different countries pursue different inflation rates, the conventional inflation target is around 2% for the advanced economies.

However, since the late 1990s when Japan fell into the liquidity trap with binding ZLB, economists, such as Krugman (1998), have debated whether central banks should raise their inflation targets above the status quo of 2% and what the optimal inflation target should be. These topics are even more important today as the US target federal funds rate has reached the zero bound since December 2008 and the US economy experienced its greatest slump since the Great Depression. Prominent economists, including Blanchard et al. (2010) and Ball (2013), suggested that policymakers might consider an inflation target of around 4%.

The suggestion lies under the argument that a significantly higher inflation target results in higher expected inflation and, as a result, higher nominal interest rates. This creates leeway for the central bank to deal with a particularly adverse demand shock before the interest rate hits the ZLB. However, higher inflation is always associated with higher price adjustment costs if firms do not foresee their future optimal prices, which is always the case, and want to adjust their prices later. In addition, higher inflation might generate a larger inefficiency wedge due to larger price distortion if firms are not able to charge their prices.

This paper aims at answering the question: what the optimal inflation target is in a fully nonlinear dynamic stochastic general equilibrium (DSGE) model with an occasionally binding ZLB. To this end, I first carefully compute the risk, or the unconditional probability, of hitting the ZLB using the US interest rates data. I then calibrate the model to match the risk using a fully nonlinear method.

I find that, based on the US interest rates data, the unconditional probability of hitting the ZLB in the US would have been 16.1% had the Fed targeted the inflation rate of 2%. Calibrating the shock to match this risk and the inflation indexation found in empirical studies, I resolve the model under different inflation targets and find that the optimal inflation rate is 3.4%, which is smaller than 4% suggested by Blanchard et al. (2010) and Ball (2013), but much larger than the conventional target of 2% pursued by many advanced economies including the US.

In addition, I find that the optimal inflation target is nonlinearly associated with the risk of hitting the ZLB and the degree of inflation indexation. It increases at a decreasing rate with the risk, but at an increasing rate with the degree of inflation indexation. However, it is always greater than 2% when the unconditional probability of hitting the ZLB is greater than 2.5% or when the inflation indexation is greater than 0.5%.

Finally, I find that, based on the benchmark calibration, the LQ approach produces an optimal inflation target of 5.5%, which is much larger than 3.4% found by the fully-nonlinear approach. Because the fully-nonlinear approach captures all the nonlinearities in the model, it would generate a more accurate optimal inflation target. So, the linear-quadratic (LQ) approach substantially overestimates the true optimal inflation target.

The related literature includes Schmitt-Grohe and Uribe (2010), Billi (2011), Coibion et al. (2012), and Ngo (2014).[1] However, all these papers are different from the current paper in many aspects. In Schmitt-Grohe and Uribe (2010), the central bank is able to commit to policy plans and the unconditional probability of hitting the ZLB is 0%. They find that the optimal inflation rate is around 0%.

Ngo (2014) also finds an optimal inflation rate of 0%. However, the central bank in his

[1] There has been extensive literature on optimal inflation target without the ZLB, for example see Antinolfi et al. (2016) for more information.

model conducts discretionary monetary policy instead of a Taylor rule, as in this paper. In addition, Ngo (2014) does not allow inflation indexation in his model, and the unconditional probability of hitting the ZLB is much lower than the probability we observed in the US data. Moreover, he models the Calvo price adjustment instead of the Rotemberg price adjustment as in this paper.

Coibion et al. (2012) find that the optimal inflation rate is around 1.5% in their benchmark model. They calibrate the model such that the unconditional probability of hitting the ZLB is about 5% post World War II, or approximately three years out of sixty years. This unconditional probability is much smaller than what I compute in this paper. The reason is that, to compute the risk of hitting the ZLB, Coibion et al. (2012) use a shorter time series of federal funds target rates. In particular, they assume that the ZLB would end in 2011. In addition, they do not address potential biases that might arise when using the nominal interest to calibrate the risk of hitting the ZLB.

Moreover, I allow a higher degree of inflation indexation that matches empirical studies, and I use the Rotemberg pricing scheme instead of the Calvo pricing scheme as in their paper. Note that Rotemberg pricing scheme together with the symmetric equilibrium assumption would eliminate the relative price distortion. Therefore, it would lower the cost of higher steady state inflation.

Billi (2011) finds that, in the case of a Taylor rule, the optimal inflation target is about 8%, which is much larger than the optimal inflation rate found in this paper. In his paper, Billi (2011) also allows inflation indexation. However, he uses the past inflation as a proxy for the expected inflation. In this paper, I use the inflation target as the proxy for the expected inflation.[2] As a result, deflationary/disinflationary episodes are less persistent in

[2]Empirical studies, such as Ascari et al. (2011), find that firms use both the past inflation and the inflation target to index their prices. It is ideal to investigate both of them at the same time. However, to improve numerical efficiency, I decide to use only the inflation target for inflation indexation. Studying

my model, and the central bank pursues a smaller inflation target.

Another key difference between this paper and the other two papers, Billi (2011) and Coibion et al. (2012), is that I solve a fully nonlinear model instead of using an LQ approximation of the model.[3] This plays an important role in creating the wedge between my results and theirs. In particular, when I use the LQ approach, the optimal inflation target is found to be 5.5%, which is closer to the value found by Billi but much larger than 3.4% found by the fully-nonlinear method. Also, the unconditional probability of hitting the ZLB declines much slower in the LQ model than in the fully-nonlinear model when the inflation target increases. Moreover, the fully-nonlinear method generates much larger expected disinflation when the risk of hitting the ZLB is high.

The remainder of this paper is organized as follows: Section 2 presents the structure of the model; Section 3 shows the solution method and the benchmark calibration, especially how to compute the risk of hitting the ZLB using the US data; Section 4 presents main results; Section 5 contains sensitivity analyses of the main findings with respect to some important parameters; Section 6 concludes; Appendices are in Section 7.

2 Model

The model in this paper is the conventional stochastic dynamic New Keynesian (DNK) model featuring the Rotemberg price adjustment.

both the past inflation and the inflation target for inflation indexation would be interesting and is part of my future research agenda.

[3]To remind the reader, according to the LQ approach, all equilibrium conditions are log-linearized, while the objective function is approximated using the second order Taylor expansion.

2.1 Households

The representative household maximizes his expected discounted utility

$$Max\ E_t \left\{ \left(\frac{C_t^{1-\gamma}}{1-\gamma} + \chi \frac{N_t^{1+\eta}}{1+\eta} \right) + \sum_{j=1}^{\infty} \left\{ \beta^j \left(\prod_{k=0}^{j-1} \beta_{t+k} \right) \left(\frac{C_{t+j}^{1-\gamma}}{1-\gamma} + \chi \frac{N_{t+j}^{1+\eta}}{1+\eta} \right) \right\} \right\} \tag{1}$$

subject to the budget constraint

$$C_t + B_t = W_t N_t + B_{t-1} \left(\frac{1+i_{t-1}}{1+\pi_t} \right) + \int_0^1 D_t(i) di + T_t, \tag{2}$$

where C, N are composite consumption and total labor; B, D, T denote real bonds, dividends, and lump sum transfers; i, π are the nominal interest rate and the inflation rate, respectively; W is the real wage; γ, η, χ are the risk aversion parameter, the inverse wage elasticity of labor with respect to wages, and the steady state labor determining parameter; β_t is the shock to the subjective time discount factor β, or the preference shock, that follows an AR(1) process

$$\ln\left(\beta_{t+1}\right) = \rho_\beta \ln\left(\beta_t\right) + \varepsilon_{\beta,t+1}, \text{ where } \beta_t \text{ is given,} \tag{3}$$

where $\rho_\beta \in (0,1)$ is the persistence of the preference shock; and $\varepsilon_{\beta t}$ is the innovation of the preference shock with mean 0 and variance σ_β^2. The preference shock is the only shock in the model.[4] It is a reduced form of more realistic forces that drive the nominal interest rate to the ZLB.[5]

[4]The preference shock is considered as the main force driving the economy to the ZLB, not the technology shock. In order for the nominal interest rate to reach the ZLB, we would need a particularly positive technology shock that we could not observe in the last recession, see Amano and Shukayev (2012) for more discussion.

[5]Another way make the ZLB binding is to introduce a deleveraging shock, see Eggertsson and Krugman (2012), Guerrieri and Lorenzoni (2011), and Ngo (2015) for more detail. Moreover, if you look at the Euler

The optimal choices of the household give rise to the implicit labor supply

$$\chi N_t^{\eta} C_t^{\gamma} = W_t, \tag{4}$$

and the Euler equation

$$E_t \left(M_{t,t+1} \left(\frac{1+i_t}{1+\pi_{t+1}} \right) \right) = 1, \tag{5}$$

where $\pi_t = P_t / P_{t-1} - 1$ is the inflation rate, and the stochastic discount factor is given by

$$M_{t,t+1} = \beta \beta_t \left(\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \right). \tag{6}$$

2.2 Final goods producers

There is a mass 1 of competitive final goods producers. To produce the composite final goods, they buy and aggregate a variety of intermediate goods using a CES technology. Their cost-minimization problem is given below.

$$\min \int_0^1 P_t\left(i\right) Y_t\left(i\right) di \text{ s.t. } Y_t = \left(\int_0^1 Y_t\left(i\right)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}, \tag{7}$$

where $P_t\left(i\right)$ and $Y_t\left(i\right)$ are the price and the amount of intermediate goods $i \in [0,1]$; and ε is the elasticity of substitution among intermediate goods.

The optimal condition gives rise to the demand for the intermediate goods i

$$Y_t\left(i\right) = \left(\frac{P_t\left(i\right)}{P_t} \right)^{-\epsilon} Y_t, \tag{8}$$

equation, the preference shock is similar to the risk premium shock, or the shock to the wedge between the borrowing and the lending rates.

and the aggregate price level

$$P_t = \left(\int P_t\left(i\right)^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}. \tag{9}$$

2.3 Intermediate goods producers

There is a mass 1 of intermediate goods producers that are monopolistic competitors. Given its price $P_t(i)$ and demand $Y_t(i)$, firm $i \in [0,1]$ chooses labor that

$$\min \left\{ W_t N_t(i) \right\} \qquad s.t. \qquad Y_t(i) = N_t, \tag{10}$$

where W_t is the economy-wide real wage rate.

Let $mc_{i,t}$ be the Lagrange multiplier with respect to the production. The first-order condition gives the same marginal cost, mc_t, to all firms:

$$mc_t = mc_{i,t} = W_t. \tag{11}$$

2.4 Price adjustments

The intermediate goods firms adjust their prices according to Rotemberg (1982). Specifically, they have to pay an adjustment cost in terms of final goods when they change their prices. Following Aruoba and Schorfheide (2013) and allowing some degree of inflation indexation, the problem of firm i, for $i \in [0,1]$, is given as follows:

$$\max_{\{P_t(i)\}} E_t \sum_{j=0}^{\infty} \left\{ M_{t,t+j} \left[\left(\frac{P_{t+j}(i)}{P_{t+j}} - mc_t \right) Y_{t+j}(i) - \frac{\psi}{2} \left(\frac{P_{t+j}(i)}{P_{t+j-1}(i)} - (1+\theta\pi) \right)^2 Y_{t+j} \right] \right\} \tag{12}$$

subject to its demand in equation (8) and

$$M_{t,t+j} = 1 \text{ if } j = 0; M_{t,t+j} = \prod_{s=0}^{j-1} M_{t+s,t+s+1} \text{ for } j \geq 1, \tag{13}$$

where ψ is the adjustment cost parameter, π is the inflation target, and θ is inflation indexation. According to this formulation, the firm is allowed to index its price to the steady state inflation, which is the same as the inflation target set by the central bank, and it only pays adjustment costs if its new price is different from the indexed price, $(1 + \theta\pi)P_{i,t-1}$.

The optimal pricing rule by firm i is given below:

$$\begin{aligned} 0 \quad = \quad & \frac{P_t(i)}{P_t} Y_t(i) - \varepsilon \left(\frac{P_t(i)}{P_t} - W_t \right) Y_t(i) - \psi \left(\frac{P_t(i)}{P_{t-1}(i)} - (1+\theta\pi) \right) \frac{P_t(i)}{P_{t-1}(i)} Y_t \\ & + \psi \beta \beta_t E_t \left(\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \left(\frac{P_{t+1}(i)}{P_t(i)} - (1+\theta\pi) \right) \frac{P_{t+1}(i)}{P_t(i)} Y_{t+1} \right) \end{aligned} \tag{14}$$

In a symmetric equilibrium, all firms will choose the same price and produce the same quantity, i.e. $P_t(i) = P_t$ and $Y_t(i) = Y_t$.[6] The optimal pricing rule then gives rise to the following condition:

$$(1 - \varepsilon + \varepsilon W_t - \psi(\pi_t - \theta\pi)(1 + \pi_t)) Y_t + \psi E_t \left[M_{t,t+1} (\pi_{t+1} - \theta\pi)(1 + \pi_{t+1}) Y_{t+1} \right] = 0, \tag{15}$$

where $M_{t,t+1}$ is the stochastic discount factor defined in equation (6).

[6] As in the existing ZLB literature, in this paper I only study a symmetric equilibrium, not other equilibria.

2.5 Monetary policy

The central bank conducts monetary policy using a simple Taylor rule as follows:

$$\left(\frac{1+i_t}{1+i}\right) = \left(\frac{Y_t}{Y}\right)^{\phi_y} \left(\frac{1+\pi_t}{1+\pi}\right)^{\phi_\pi}, \tag{16}$$

$$i_t \geq 0, \tag{17}$$

where π, i, Y are the inflation target, the steady state nominal interest rate, and the steady state output, respectively.

Equation (17) implies that the nominal interest rate is not allowed to be negative. This is the key condition in the ZLB literature.

2.6 Aggregate conditions and equilibrium

The aggregate output is

$$Y_t = N_t, \tag{18}$$

and the resource constraint is given by

$$C_t + \frac{\psi}{2}\left(\pi_t - \theta\pi\right)^2 Y_t = Y_t. \tag{19}$$

The equilibrium system consists of six nonlinear difference equations (4), (5), (15), (16), (18), (19) together with the ZLB (17) for six variables W_t, C_t, i_t, π_t, N_t, and Y_t.[7]

[7]See the appendix for the fully-nonlinear equilibrium system. In addition, the appendix also shows the LQ version of this model.

3 Calibration and solution method

3.1 Calibration

3.1.1 The risk of hitting the ZLB

The risk of hitting ZLB plays a key role in determining the optimal inflation target. Specifically, when the risk of hitting the ZLB is high, the expected cost of hitting the ZLB is high given the fact that the ZLB is very damaging. Many economists, including Mishkin (2011), believe that the 2007-2009 recession with a binding ZLB is a rare disaster that occurs once every seventy years. However, other economists, i.e. Ball (2013), disagree. In this section, I carefully compute the risk that the nominal interest rate hits the ZLB if the Fed continues to keep the inflation target at 2%.

Panel A of Figure 1 shows the target and effective federal funds rates in the US since 1981:I.[8] These rates are tightly correlated and they have reached the ZLB since December 2008 when the economy was in the middle of the 2007-2009 recession.[9] By the end of 2014, the ZLB duration had been 25 quarters, which is much longer than the existing ZLB literature predicted.

Ideally, one should use the target federal fund rate to compute the unconditional probability for it to hit the ZLB. Using this time series, the unconditional probability of hitting the ZLB is computed to be 19.2%, or 25 quarters out of 130 quarters. This unconditional probability might overstate the actual risk of hitting the ZLB in the US because the series is relatively short.

Given the fact the the target and the effective federal funds rates are closely related

[8]The data for the target federal funds rate was not available until 1982:III.

[9]To be more precise, the effective federal funds rate has been very close to zero, around 0.12% on average, during this period.

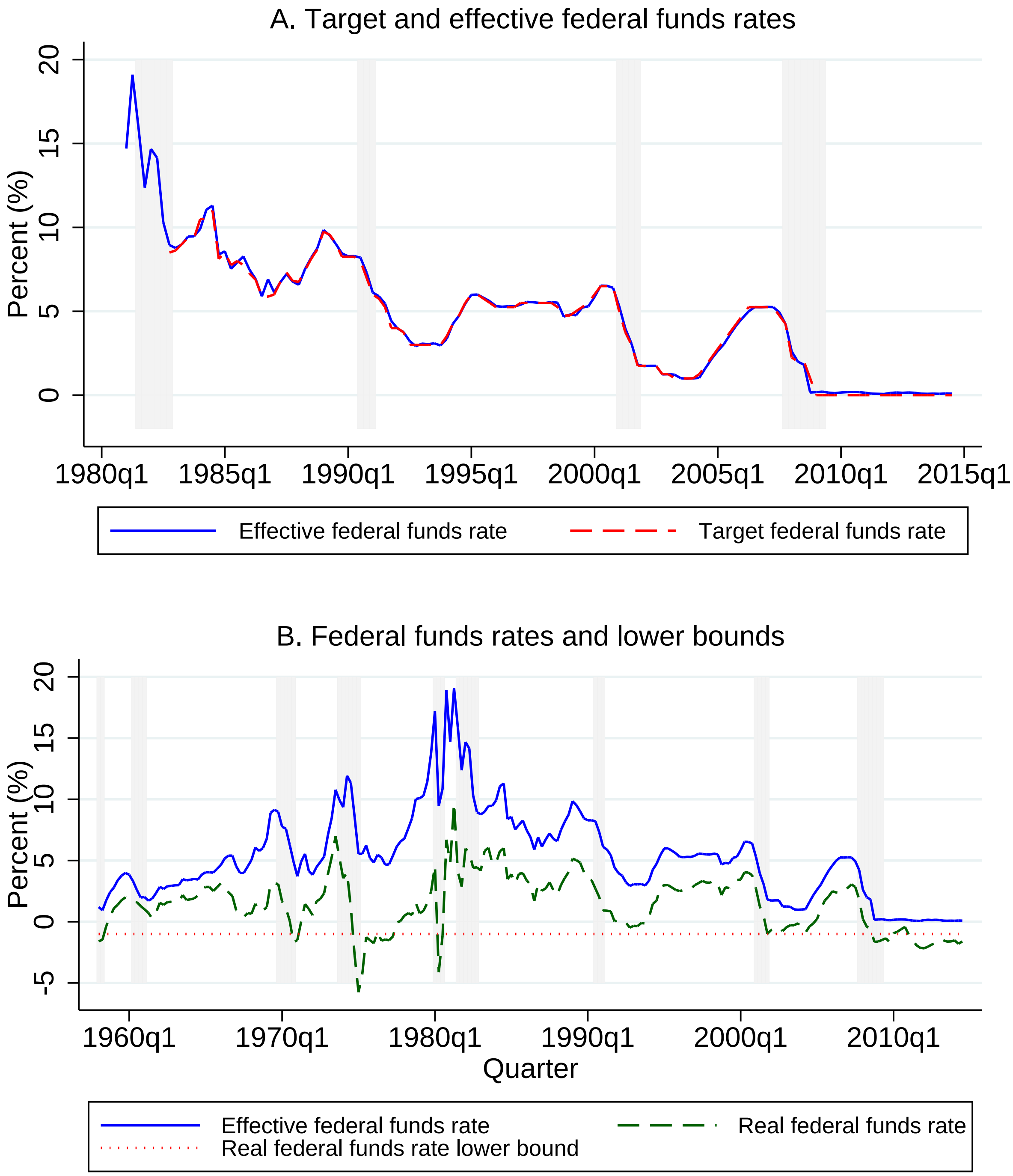

Figure 1: Real federal funds rate is the effective federal funds rate minus the inflation rate computed as a percentage change in the CPI of All Items Less Food and Energy a year ago. The shaded areas indicate the US recessions. Source: the Federal Reserve Economic Data.

as seen in Panel A of Figure 1, we can use the effective federal funds rate to have more observations, 244 quarters starting from June 1954. Based on this data, the unconditional probability of hitting the ZLB is 10.3%. However, this number might understate the actual risk of hitting the ZLB because of the fact that, not until the very early 1990s, the Fed and other central banks started pursuing the inflation target of 2%, either explicitly or implicitly.

To address these biases, we should answer the question raised in Ball (2013): what would the unconditional probability of hitting the ZLB have been, had the Fed targeted the inflation rate of 2%. To this end, I follow Ball (2013) and use the real interest rate to answer the question. Specifically, the nominal interest rate equals the real interest rate plus the expected inflation rate. Therefore, we can interpret the zero lower bound on the nominal interest rate as a bound of minus expected inflation for the real interest rate. If the target inflation rate is 2%, the expected inflation rate would be 2% and the bound on the real interest rate would be -2%. However, Ball (2013) argues that a recession is likely to push expected inflation down somewhat and that the history suggests that the inflation fell about 1% during the past recessions that started with $2-3\%$ inflation rates. Therefore, he finds that the bound on the real interest rate is -1%.

Panel B of Figure 1 shows: (i) the effective federal funds rate; (ii) the real interest rate computed as the effective federal funds rate minus the inflation rate, where the inflation rate is calculated as a percentage change of the CPI of All Items Less Food and Energy from a year ago; and (iii) the lower bound of the real interest rate. The data spans from 1957:IV, when the data for the CPI of All Items Less Food and Energy was first available, to 2014:IV. So, we have 229 observations in all.

From Panel B, we are able to see that the real interest rate was smaller than the bound, and, as a result, the nominal interest rate might have hit the ZLB, in the five recessions: 1957:III-1958:II, 1969:IV-1970:IV, 1973:IV-1975:I, 1980:I-1980:IV, and 2007:IV-2009:II.[10] Es-

[10]Ball (2013) argues that in the three out of seven recent recessions excluding the 2007-2009 recession,

pecially, using the real interest rate, we can very well infer that the nominal interest rate reached the ZLB during the 2007-2009 recession. In addition, the nominal interest rate almost hit the ZLB in the 2001 recession.

Examining the real interest rate since 1957:IV when the CPI data was first available, I find that the ZLB was binding in 37 quarters. Given that the sample has 229 quarters, the unconditional probability of hitting the ZLB is 16.1%. [11] This value is greater than any value used in the existing ZLB literature, including Coibion et al. (2012). For example, they calibrate the model such that the unconditional probability of hitting the ZLB is three years out of sixty years, or around 5%, if the inflation target is 3.5%. According to this calibration, the unconditional probability of hitting the ZLB would be only 10% if the inflation target was 2%.

However, the unconditional probability of 16.1% is potentially biased due to the following implicit assumptions. First, the Fed always targeted 2%, and second, there was a credible commitment to a 2% inflation target. This is obviously not true in reality. Therefore, in Section 5 - Sensitivity analysis, I recalibrate the shock, resolve the model, and present results

the nominal interest rate would have hit the ZLB if the inflation rate had been around 2% at the start of the recessions. These three recessions include the 1969-1970 recession, the 1973-1975 recession, and the 1980 recession. Hence, the probability of hitting the ZLB conditional on a recession would be around 50%, or four recessions out of eight recessions, if the Fed targeted 2% inflation rate post World World II.

[11] For a robust check, I also used the CPI of All Items and the PCE index, instead of the CPI of All Items Less Food and Energy. The result slightly changes. However, it is still in the range from 15% to 17%. Another robust check is to raise the lower bound on the real interest rate. The reason is that, using the real interest rate slightly underestimates the actual ZLB as the real interest rate was larger than the bound in 2009:IV and early 2010. This occurs because the inflation rate fell more than 1%, the decrease that Ball (2013) assumes in order to find out the bound for the real interest rate. If we allowed the inflation rate to fall more than 1% in a recession, i.e. 1.5%, then the lower bound on the real interest rate would be -0.5% if the Fed targeted a 2% inflation rate. Using this new lower bound, the risk of hitting the ZLB would increase to 20.6%. However, the ZLB was binding in many periods when the economy was in normal time.

for different values of probability, ranging from 0% to 20%.

3.1.2 Parameterization

I calibrate the parameters on the basis of the observed data and other studies. The quarterly subjective discount factor β is set at 0.997, as in Woodford (2011). The constant relative risk aversion parameter γ is 1, corresponding to a log utility function with respect to consumption. This utility function is commonly used in the business cycles literature. The labor supply elasticity with respect to wages is set at 1, or $\eta = 1$, as in Woodford (2011). I set the parameter associated with labor preference $\chi = 1$.[12] The elasticity of substitution among differentiated intermediate goods ϵ is 7.66, corresponding to a 15% net markup. This value is also popular in the literature (e.g., Adam and Billi (2007) and Braun et al. (2013)).

The price adjustment cost parameter, ψ, is calibrated to be 132 corresponding to the probability of keeping prices unchanged of 0.8, which is in the range estimated by Christiano et al. (2005) and is still smaller than the value used in Christiano et al. (2011), 0.85.[13] Some authors, including Nakata (2011) and Ascari et al. (2011), use/estimate higher values for the price adjustment cost parameter. However, using these higher values would result in a longer duration of keeping prices unchanged that is not in line with empirical studies, see Nakamura and Steinsson (2008) for more detailed discussion.

[12] As in Fernandez-Villaverde et al. (2012), this parameter does not affect the results of the paper significantly.

[13] This comparison is up to the first-order approximation around the steady state inflation of 0%, see Miao and Ngo (2014) for more discussion.

Table 1. Benchmark Parameterization

Symbol	*Description*	*Values*
β	Quarterly discount factor	0.997
γ	CRRA parameter	1
η	Inverse labor supply elasticity	1
ε	Monopoly power	7.66
θ	Inflation indexation	0.9
ψ	Price adjustment cost parameter in the Rotemberg model	132
π	Inflation target, 2% per year	0.005
ϕ_π	Weight of target inflation in the Taylor rule	1.5
ϕ_y	Weight of output target in the Taylor rule	0.125
ρ_β	AR-coefficient of preference shocks	0.65
σ_β	Standard deviation of the innovation of preference shocks (%)	0.5

I set the parameters in the Taylor rule $\phi_\pi = 1.5$ and $\phi_y = 0.125$, as in Gali (2008) and Ascari and Rossi (2012), which are conventional in the literature. I set the inflation indexation, θ, at 0.9, which is in the range estimated by Ascari et al. (2011) and is the same degree of inflation indexation used in Billi (2011). However, this value is controversial. For example, Cogley and Sbordone (2008) find no evidence for price indexation. Therefore, I also conduct some sensitivity analysis with respect to this parameter in Section 5.

Following Nakov (2008), I set the persistence of the preference shock, ρ_β, at 0.65, which reflects the persistence of the natural rate of interest rate. Nakov (2008) argues that this value is between 0.35 used by Woodford (2003) and 0.8 used by Adam and Billi (2007). The remaining and the most difficult task is to determine how large the standard deviation of the innovation of preference shock is.

In this paper, I calibrate the standard deviation of the innovation of preference shocks to be 0.5% per quarter, which enables the model to generate the unconditional probability of hitting the ZLB of 16.1%, based on the fully-nonlinear method that I describe in the section below. However, as I discussed previously, this unconditional probability of hitting the ZLB is controversial. Therefore, I also implement a sensitivity analysis for the results of this paper regarding the risk of hitting the ZLB in Section 5.

3.2 Solution method

Following the method used in Ngo (2014), I solve the model using a collocation method associated with cubic spline basis functions to capture kinks due to the ZLB. At each collocation node, I solve a complementarity problem using the Newton method and the semi-smooth root-finding algorithm as described in Miranda and Fackler (2002). I also use an analytical Jacobian matrix computed from the approximating functions. Moreover, I write the code using a parallel computing method that allows me to split up a large number of collocation nodes into smaller groups that are then assigned to different processors to be solved simultaneously. This procedure reduces computation time significantly.[14]

Multiple equilibria is well-known in the existing ZLB literature, mainly in the setting where the ZLB exogenously follows a two-state Markov chain with non-ZLB being the absorbing state. In particular, the economy is assumed to initially stay at the ZLB and it will escape from the ZLB with an exogenous probability; after escaping from the ZLB, the economy will stay out of it forever. Braun et al. (2013) show that there are multiple equilibria at the ZLB in this setting. Aruoba and Schorfheide (2013) also find this issue. However, this multiple-equilibria issue does not occur in the stochastic setting as described in this paper,

[14] I obtain the maximal absolute residual across the equilibrium conditions of the order of 10^{-8} for almost all states off the collocation nodes. For a few off-collocation states when the ZLB becomes binding, the maximal absolute residual is of the order of 10^{-5}.

see Richter and Throckmorton (2015) for further discussion.

4 Results

To find out the optimal inflation target, I implement the following procedure. For each inflation target, I first solve the model to find the value and policy as functions of the preference shock. I then take a random sample of $99,999$ preference shocks, and compute the unconditional (average) welfare based on the value function found in the first step. Eventually, I compute welfare gain as a percentage change of the unconditional welfare from the one associated with the conventional inflation target of 2% that the Fed has pursued implicitly. In addition, I compute the long-run inflation, which is the average inflation rate from the simulation of $99,999$ periods. I also compute the unconditional probability of hitting the ZLB based on the $99,999$-period simulation, which is the ratio between the number of periods with binding ZLB and $99,999$ periods simulated.

I plot the welfare gain, the long-run inflation, and the unconditional probability against inflation targets in Figure 2. The solid red line in Figure 2, with the y-axis on the right, presents the unconditional welfare gain. Apparently, the welfare gain is a nonlinear function with respect to the inflation target. It first increases with the inflation target and reaches the highest value at the inflation target of 3.4% before decreasing with the inflation target.

The intuition for the nonlinear relationship is that: when the inflation target is higher, both the probability of hitting the ZLB and output loss are smaller. As a result, the benefit of targeting an inflation rate higher than 2% is more than to offset the cost caused by the higher inflation target. Therefore, the welfare gain increases. However, when the inflation target is much higher than 2%, the cost of the higher inflation target outweighs the benefit, and the welfare gain decreases. In conclusion, the optimal target inflation rate is 3.4% per year.

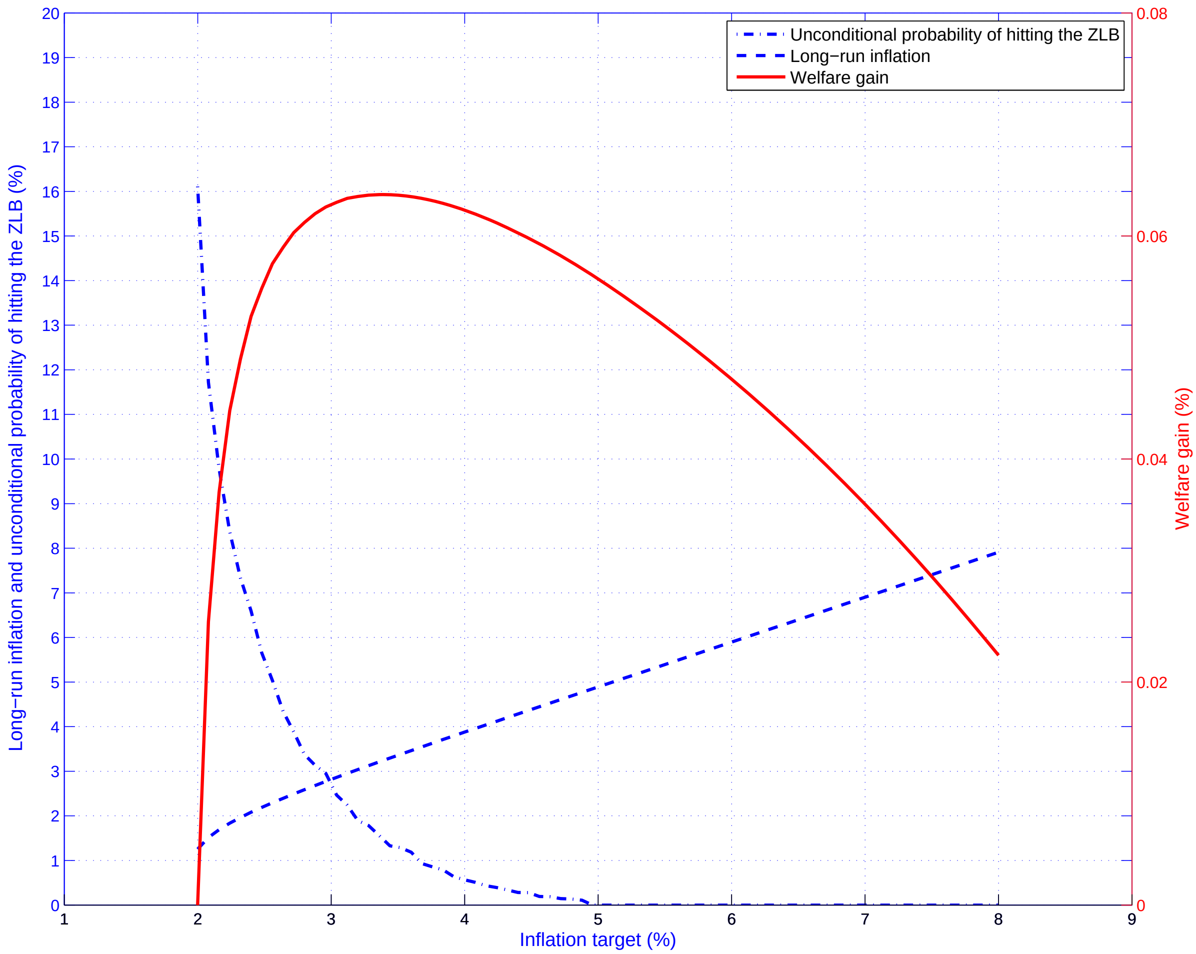

Figure 2: Welfare gain, long-run inflation, and unconditional probability of hitting the ZLB. Welfare gain is a percentage change in unconditional welfare due to targeting an inflation rate greater than the conventional target of 2%. Unconditional welfare is the average welfare based on a sample of 99,999 preference shocks. Long-run inflation is the average inflation rate based on the simulation of 99,999 periods. Unconditional probability of hitting the ZLB is the ratio between the number of periods with a binding ZLB and 99,999 periods simulated.

As shown in Figure 2 with the y-axis on the left, the higher the inflation target, the higher the long-run inflation, and the lower the probability of hitting the ZLB. For example, if the inflation target is 2%, the long-run inflation is 1.2% and the probability of hitting the ZLB is around 16.1%. When the inflation target increases to 3.5%, the long-run inflation is 3.5% and the unconditional probability of hitting the ZLB reduces to around 1.3%. It is interesting to note that, even the inflation target is 2%, the average inflation is only 1.2%, which is very much in line with what we actually observed in the reality. This is the well-known disinflationary bias.

The optimal inflation target of 3.4% is between the ones found in Coibion et al. (2012) and Billi (2011). Specifically, Billi (2011) finds that, in the case of a Taylor rule, the optimal inflation target is as big as about 8%. The main reason for the difference is that Billi (2011) uses the last inflation for inflation indexation. Due to this characteristic, under a particularly adverse shock that causes the ZLB to bind, his model generates very persistent deflationary/disinflationary episodes associated with binding ZLB and output loss. Consequently, a binding ZLB is very damaging in his model. Hence, the central bank pursues a very high inflation target. In his framework, the probability of hitting the ZLB is zero under the optimal inflation target of 8%.

Instead of using the past inflation as the benchmark for inflation indexation as in Billi (2011), in this paper I use the inflation target for indexing inflation. By doing so, the model does not produce very persistent deflationary/disinflationary episodes associated with binding ZLB. Therefore, a binding ZLB is less damaging in this paper than in Billi (2011), and the optimal inflation target is smaller than the one found in his paper. Note that the degree of inflation indexation of 0.9 in this paper is similar to the one used in Billi (2011).

Another important reason is due to inaccuracy of the LQ approach. As shown in the sensitivity analysis below, the LQ approach substantially overestimates the true optimal inflation target that it would be under the fully nonlinear method as in this paper. This is due to the fact that the fully-nonlinear method is able to capture all nonlinearity of the model, while the LQ approach is not.

The optimal inflation target in the benchmark model of Coibion et al. (2012) is around 1.5% per year and smaller than the one in this paper. There are several reasons for the discrepancy. First, I model the price adjustment using adjustment costs, as in Rotemberg (1982), instead of using the time-dependent pricing, as in Calvo (1983). This is a very important reason because in the Rotemberg model, the assumption of symmetric equilibrium is commonly used. As a result, the relative price dispersion is zero regardless of inflation targets, and this helps to lower the cost of higher steady state inflation. To illustrate this point, I plot steady state welfare gains against the inflation target in the models with Rotemberg and Calvo pricing schemes, Panel A of Figure 3. Steady state welfare/consumption gains are computed as percentage changes in steady state welfare/consumption from the ones associated with zero inflation target. The inefficiency wedge in the Rotermberg model is the price adjustment cost, as percentage of output, while it is the relative price dispersion in the Calvo model.

The panel shows that the welfare loss increases faster in the Calvo model than in the Rotemberg model when the inflation target increases. So, the benefit of having a higher inflation target may not enough to offset the loss in the Calvo model. As a result, the Calvo model produces a smaller optimal inflation target.

Second and equally importantly, I calibrate the preference shock such that the unconditional probability of hitting the ZLB matches what we observed in the US post World War II. This probability is much higher than the one used in the benchmark model of Coibion et al. (2012), making both the conditional and unconditional cost of the ZLB higher.

In addition, the degree of inflation indexation in this paper is 0.9, which is in the range estimated by Ascari et al. (2011) and is the same value used in Billi (2011). This degree of inflation indexation is greater than the one used in Coibion et al. (2012). As a result, the

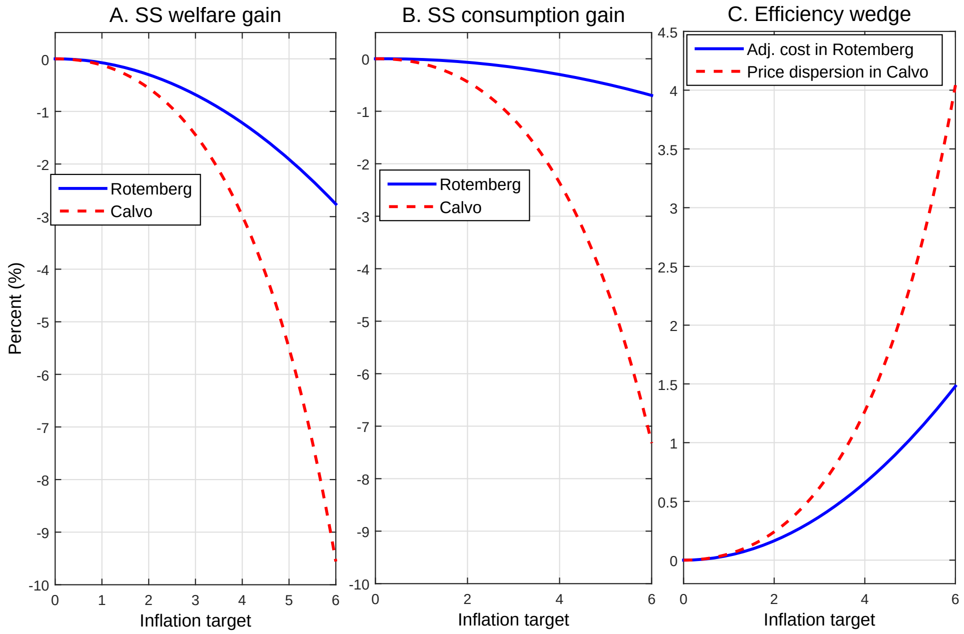

Figure 3: Steady state welfare gain, steady state consumption gain, and inefficiency wedge.

cost of high steady state inflation incurred every period is smaller in this paper than in their paper. Even though their LQ approach overestimates the optimal inflation target relative to the fully-nonlinear method, the net benefit of targeting a highly positive inflation rate is still greater in this paper.

I will conduct sensitivity analyses of the result of this paper with respect to alternative numerical methods used to solve the model, and with important parameters, including the risk of hitting the ZLB, the inflation indexation, the adjustment cost parameter, the persistence of the innovation of the preference shock in the following section. So, we are able to see how the optimal inflation target changes with these parameters in the next section.

5 Sensitivity analyses

5.1 Fully-nonlinear versus linear-quadratic (LQ) approaches

In this subsection, I discuss the difference between results from the benchmark fully nonlinear model and those from the LQ version of this model.

It is important to note that if I use the benchmark calibration, the LQ model produces the unconditional probability of hitting the ZLB of only 0.25% instead of 16% as in the benchmark fully-nonlinear model, and the unconditional welfare of the LQ model monotonically decreases in inflation target starting at 2%.

To make the two models more comparable, I first recalibrate the variance of the preference shock innovation from 0.5% to 1.33%. With this new value, the LQ produces an unconditional probability of 16% given the inflation target of 2%. Then I solve the LQ model under different inflation targets, and plot results in Figure 4, presented by the dashed blue lines.

Surprisingly, as seen in Panel A, the unconditional welfare of the LQ model increases and peaks at the inflation target of 5.5%, much higher than the optimal target of 3.4% as in the benchmark fully-nonlinear model. The reason is that when the target inflation increases, the frequency of hitting the ZLB decreases at a smaller rate in the LQ model than in the fully-nonlinear model, as shown in Panel B. As a result, the risk of hitting the ZLB is higher in the LQ model, and it is optimal for the central bank to pursue a higher inflation target.

To provide further explanation for the discrepancy, I also solve the linear-nonlinear model, where I log-linearize the equilibrium conditions as in the LQ approach. However, instead of approximating the objective function using the quadratic approach as in the LQ model, I use the exact objective function. The result from the linear-nonlinear model is presented by the dot-dashed green lines. We can easily see that the result from the LQ model and the linear-nonlinear are very close. They both indicate that the optimal inflation target is 5.5%.

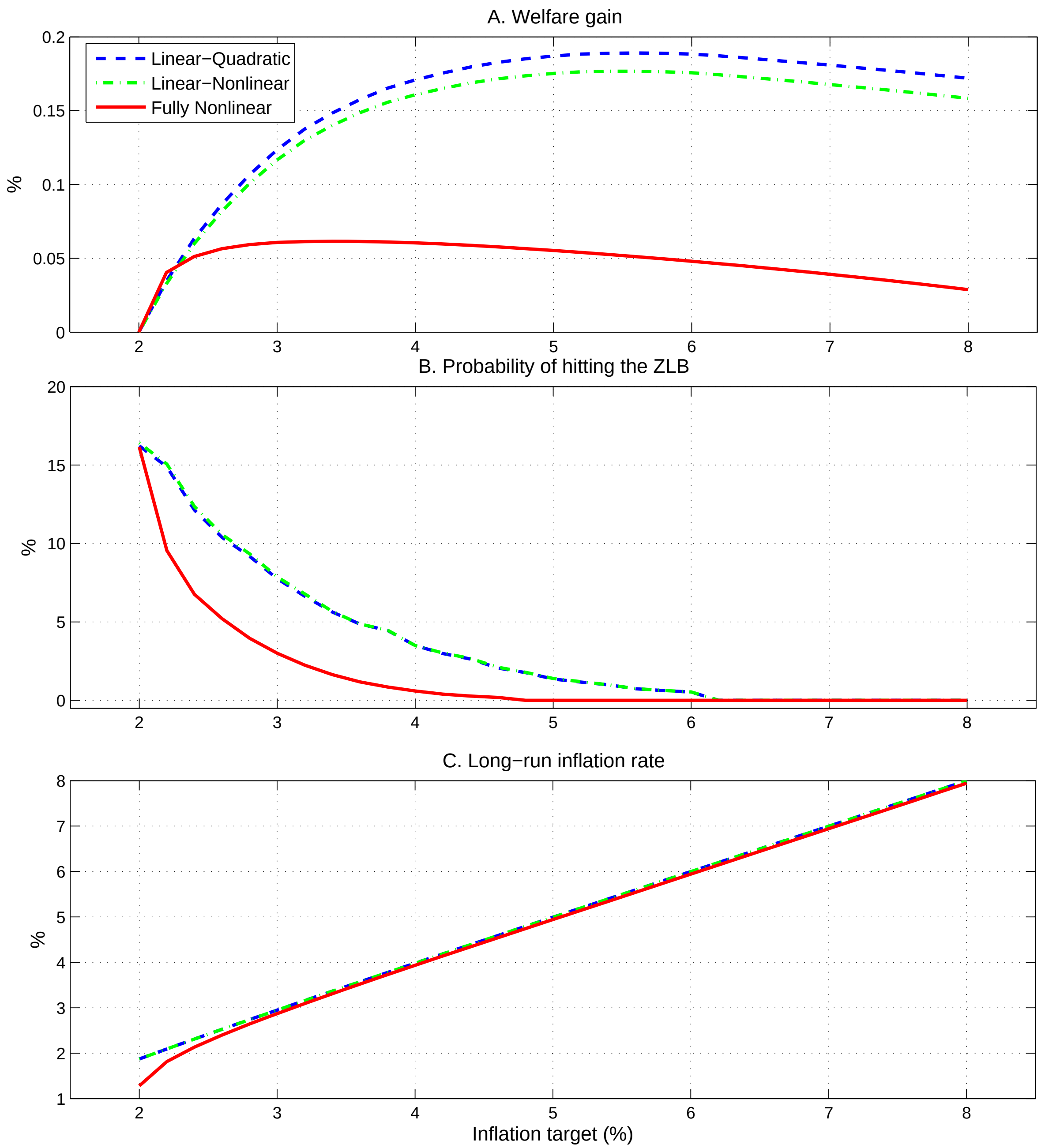

Figure 4: Comparison between the fully-nonlinear method and the LQ approach. Welfare gain is a percentage change in unconditional welfare due to targeting an inflation rate greater than the conventional target of 2%. Unconditional welfare is the average welfare based on a sample of 99,999 preference shocks. Long-run inflation is the average inflation rate based on the simulation of 99,999 periods. Unconditional probability of hitting the ZLB is the ratio between the number of periods with binding ZLB and 99,999 periods simulated.

Therefore, the difference between the LQ and the fully-nonlinear model can not be explained by the quadratic approximation of the objective function. In fact, the discrepancy is due to the first order approximation of the equilibrium conditions, and this approximation could not capture the risk of hitting the ZLB correctly.

These differences would be useful in explaining why the optimal inflation target in Billi (2011) might be too high, and why the result in Coibion et al. (2012) might be potentially inaccurate.

5.2 The risk of hitting the ZLB

To see how sensitive the optimal inflation target is to the risk of hitting the ZLB, I plot the optimal inflation target against different values of unconditional probability of hitting the ZLB. To help the reader understand more about the result, I first explain the procedure. I then discuss the result.

To compute the optimal inflation target for each probability, I first recalibrate the volatility of the innovation of the preference shock such that the unconditional probability of hitting the ZLB matches the probability given the other parameters values. For each case, I then resolve the model and find out the optimal inflation target. The result is presented in Panel A of Figure 5.

Apparently, the optimal inflation target is positively correlated with the risk of hitting the ZLB. Intuitively, when the risk of hitting the ZLB increases, the expected cost of hitting the ZLB increases. Therefore, it is better for the central bank to raise the optimal inflation target.

It is not surprising to see the optimal inflation target rising as the risk increases. However, it is interesting to learn that the relationship is non-linear. The optimal inflation target increases at a smaller rate when the unconditional probability of hitting the ZLB increases. The optimal inflation target is 0% when there is an extremely small chance for the nominal

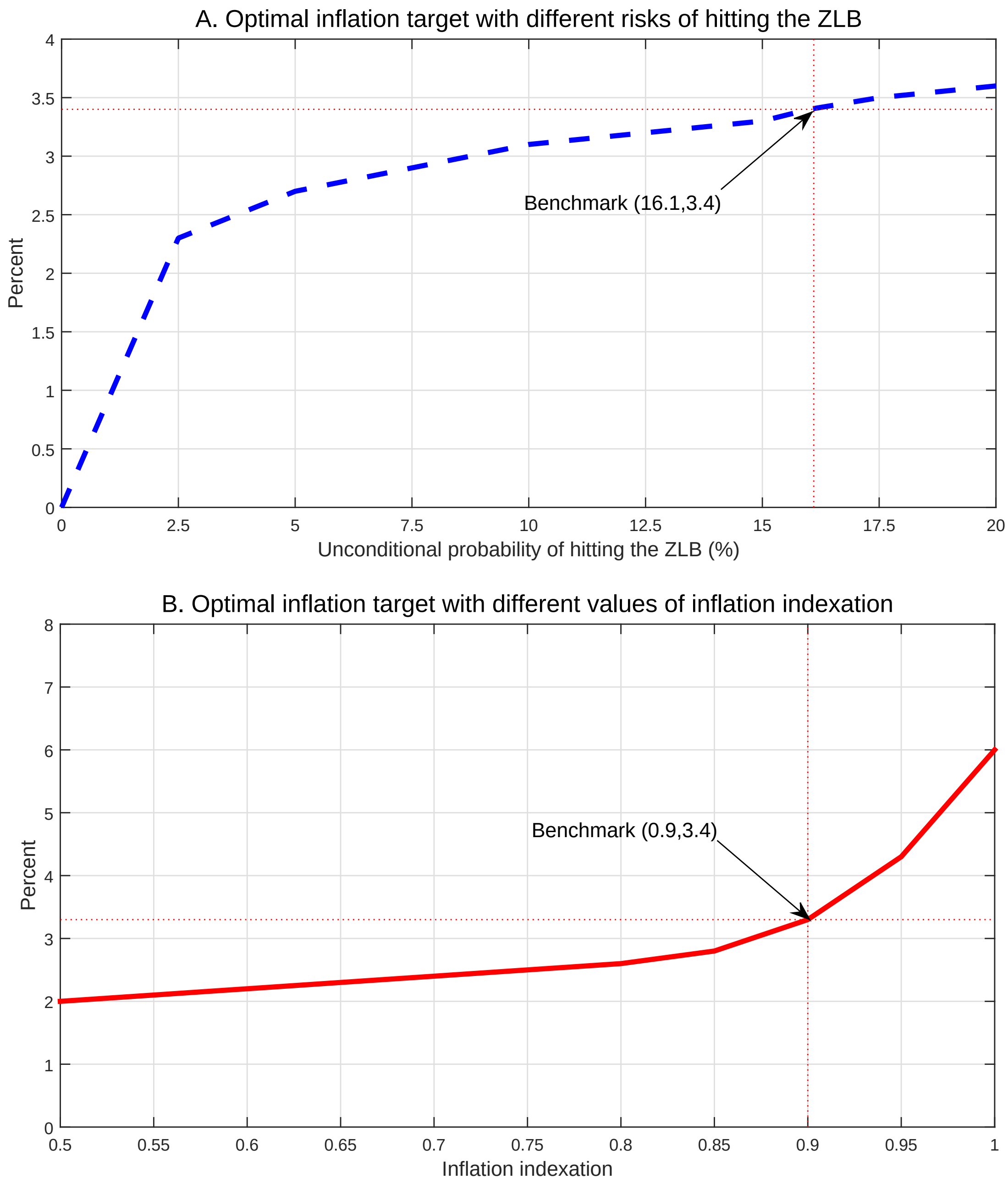

Figure 5: Optimal inflation target under different values of probability of hitting the ZLB and different values of inflation indexation. In the benchmark calibration, the unconditional probability of hitting the ZLB is 16.1% and the inflation indexation is 0.9. The optimal inflation target under the benchmark calibration is 3.4%.

rate to hit the ZLB. It increases to 2.2% when the the probability is 2.5%, and to 2.6% when the risk of hitting the ZLB is 5%. The increase in the optimal inflation target is smaller when the probability increases by another 2.5%, and so on.

In addition, it is important to know that the optimal inflation target is much higher than 2% even if the risk of hitting the ZLB is computed as low as 2.5%, and that the optimal inflation target is always greater than 3% when the risk is greater than 10%.

5.3 Inflation indexation

I compute the optimal inflation target under different values of inflation indexation as follows. With each value of inflation indexation, I first recalibrate the standard deviation of the innovation of preference shocks such that the unconditional probability of hitting the ZLB remains unchanged, around 16.1%. Afterward, I resolve the model under different inflation targets, and find out the optimal inflation target that is associated with the highest welfare. I plot the optimal inflation target against inflation indexation in Panel B of Figure 5.

Panel B of Figure 5 shows that for the benchmark calibration with 0.9 inflation indexation, the optimal target inflation is 3.4%. Overall, the optimal inflation target is positively correlated with inflation indexation. When the inflation indexation is 0.5, the optimal inflation target is around 2%. For the case when the inflation is fully indexed, the optimal target inflation is 6%. Note that when the inflation indexation is smaller than 0.5, the optimal inflation target is not greater than 2% given the unconditional probability of hitting the ZLB of 16.1%.

Intuitively, when the inflation indexation is higher, the marginal cost of raising the inflation target is smaller because firms' prices are closer to the optimal price even if the firms do not pay adjustment costs to change their prices. In the meantime, with a higher inflation target, the probability of hitting the ZLB decreases, and the risk of falling into a deep recession with a binding ZLB is smaller. Therefore, it is better for the central bank to raise

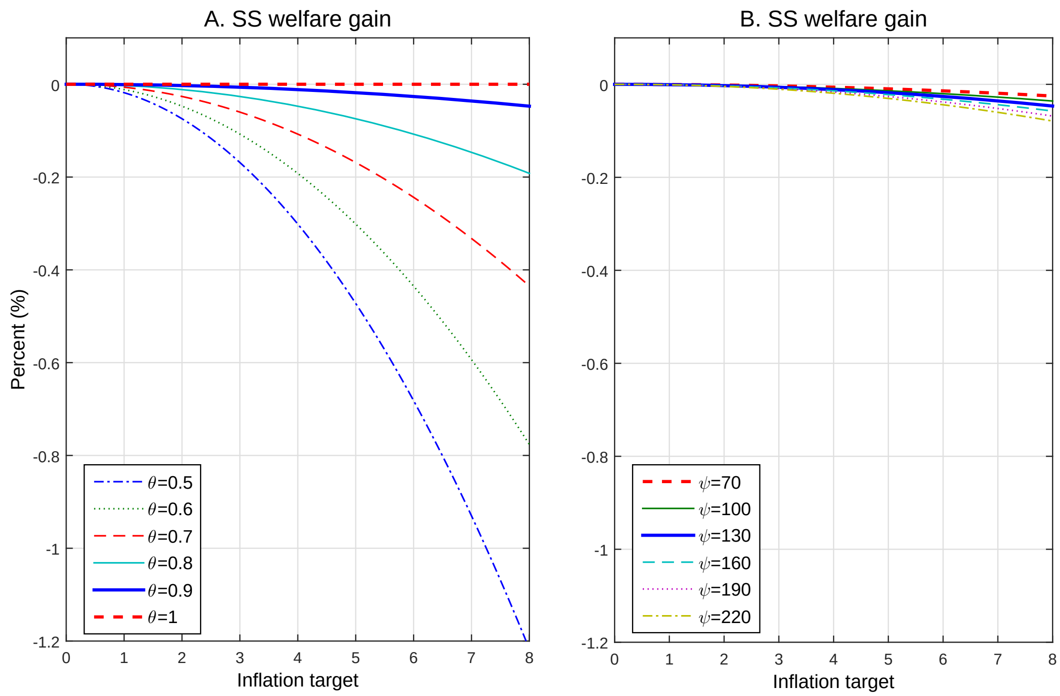

Figure 6: Steady state welfare gains under different values of adjustment cost parameter and inflation indexation.

the inflation target.

The fact that the optimal inflation target increases with inflation indexation might not be surprising. It is more interesting to see how optimal inflation target changes with inflation indexation. From Panel B of Figure 5, the relationship is nonlinear and convex. The optimal inflation target is always greater than 2% when the indexation is greater than 0.5. Then it increases gradually with inflation indexation. It only takes off when the indexation is greater than 0.85.

5.4 The adjustment cost parameter

The inefficiency wedge of the Rotemberg model depends on both the price adjustment cost parameter and the inflation indexation. In this sub-section, I will discuss the role of price adjustment cost parameter *indirectly using comparative static*. Figure 6 shows the steady state welfare gain as a function of inflation target, under different values of the adjustment cost parameter and different values of inflation indexation. The steady state welfare gain is computed as a percentage change in steady state welfare from the one associated with zero inflation target.

Apparently, in all cases, given the values of the other parameters, welfare decreases as the inflation target increases. In addition, given an inflation target, welfare decreases with the price adjustment cost and increases with inflation indexation. From Figure 6, welfare changes are much less significant with the adjustment cost parameter than with inflation indexation. Therefore, the optimal inflation target would change more with inflation indexation than with the adjustment cost parameter that is in the range investigated here.

5.5 The persistence of the preference shock

The persistence of the preference shock is interesting because it represents the expected ZLB duration. In this paper, the baseline value of 0.65 is still in the range used in the literature, see Nakov (2008). Although the expected ZLB duration corresponding to this baseline value is about 3 periods, the simulation showed that there are some recessions with 20-period ZLB duration.

Given unconditional probability of hitting the ZLB, the optimal inflation target would be higher with a more persistent shock. The intuition is that a recession would be worse with the more persistent shock. Therefore, the optimal inflation targets found in this paper would underestimate the true optimal inflation targets if a stable equilibrium exists for the case of a more persistent shock.

However, to those whose have solved an occasionally binding ZLB model, it is very common that there is a trade-off between unconditional probability of hitting the ZLB and expected ZLB duration in solving the model, see Richter and Throckmorton (2015) for more discussion. Based on my numerical method, the model does not have a stable solution if the persistence is greater than 0.7 given the unconditional probability of hitting the ZLB of more than 16%, unless you truncate the state spaces of the shock. However, truncating the space is not preferable because it distorts results substantially.

6 Conclusion

This paper investigates what inflation target is optimal in the New Keynesian framework featuring an occasionally binding ZLB on the nominal interest rate. Solving the fully nonlinear model using a global method, I find that, under the calibration that matches the unconditional probability of hitting the ZLB in the US and with the inflation indexation found in empirical studies, the optimal inflation rate is 3.4%. In addition, I find that the LQ approach substantially overestimates the true optimal inflation target found by the more accurate fully-nonlinear method.

The optimal inflation target is positively correlated with the risk of hitting the ZLB and the degree of inflation indexation. It is always greater than 2% as long as the probability of hitting the ZLB is greater than 2.5% or the inflation indexation is greater than 0.5. The relationship between the optimal inflation target and these metrics are nonlinear: it is concave in the formal, and convex in the latter.

There are several ways in which we can extend this paper. First, in this paper, firms adjust their prices using the Rotemberg pricing scheme, which is just a simple menu cost model. It would be interesting to see if the result changes under different pricing schemes, such as the Calvo time-dependent pricing scheme or the state dependent pricing scheme of Dotsey et al. (1999) or Gertler and Leahy (2008). Second, it would be interesting to see if adding more realistic features, such as habit formation, policy inertia, and inflation inertia, would change the result significantly.[15]

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[15]Studying the impact of different pricing schemes and these features in a fully-nonlinear framework as in this paper is part of my future research agenda.

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7 Appendix

7.1 The fully-nonliear model

$$\max\left\{\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi}\left(\frac{Y_t}{Y}\right)^{\phi_\pi}-\frac{R_t}{R},\frac{1}{R}-\frac{R_t}{R}\right\}=0 \tag{20}$$

$$\frac{1}{R_tC_t^\gamma}=\beta\beta_tE_t\left(\frac{1}{C_{t+1}^\gamma}\frac{1}{\Pi_{t+1}}\right) \tag{21}$$

$$\left(1-\varepsilon+\varepsilon W_t-\psi\left(\pi_t-\theta\pi\right)\Pi_t\right)\frac{Y_t}{C_t^\gamma}+\psi\beta\beta_tE_t\left[\frac{Y_{t+1}}{C_{t+1}^\gamma}\left(\pi_{t+1}-\theta\pi\right)\Pi_{t+1}\right]=0 \tag{22}$$

$$C_t=\left(1-\frac{\psi}{2}\left(\pi_t-\theta\pi\right)^2\right)Y_t \tag{23}$$

$$Y_t=N_t \tag{24}$$

$$W_t=\chi C_t^\gamma N_t^\eta \tag{25}$$

$$V_t \quad = \quad \frac{C_t^{1-\gamma}}{1-\gamma}-\chi\frac{N_t^{1+\eta}}{1+\eta}+\beta\beta_tE_t\left[V_{t+1}\right] \tag{26}$$

where $\Pi_t=1+\pi_t,\ R_t=1+i_t, V_t=V\left(\beta_t\right).$

7.2 The linear-quadratic (LQ) model

I follow Woodford (2003) to log-linearize the equilibrium conditions and approximate the objective function using a quadratic approximation.

$$\max\left\{\phi_\pi\widehat{R}_t+\phi_y\widehat{Y}_t-\widehat{R}_t,-\log\left(R\right)-\widehat{R}_t\right\}=0 \tag{27}$$

$$-\left(\widehat{R}_t+\gamma\widehat{C}_t\right)=\widehat{\beta}_t+E_t\left(-\gamma\widehat{C}_{t+1}-\widehat{\Pi}_{t+1}\right) \tag{28}$$

$$0 = \frac{\varepsilon W \widehat{W}_t - \left[\psi\left(1-\theta\right)\pi\Pi + \psi\Pi^2\right]\widehat{\Pi}_t}{\psi\beta\left(\theta-1\right)\pi\Pi} - \left(\widehat{Y}_t - \gamma\widehat{C}_t\right) + \widehat{\beta}_t \tag{29}$$

$$+\left(\widehat{Y}_{t+1} - \gamma\widehat{C}_{t+1} + \left(\frac{\Pi}{\pi\left(1-\theta\right)} + 1\right)\widehat{\Pi}_{t+1}\right) \tag{30}$$

$$\widehat{C}_t = \widehat{Y}_t - \frac{\psi\left(\pi - \theta\pi\right)\Pi Y}{C}\widehat{\Pi}_t \tag{31}$$

$$\widehat{Y}_t = \widehat{N}_t \tag{32}$$

$$\widehat{W}_t = \gamma\widehat{C}_t + \eta\widehat{N}_t \tag{33}$$

$$V_t = \left[\left(\frac{C^{1-\gamma}}{1-\gamma} - \chi\frac{N^{1+\eta}}{1+\eta}\right) - \frac{1}{2}\left(\gamma+\eta\right)C^{1-\gamma}\left(\widehat{C}_t - x^*\right)^2\right] + \beta E_t\left[V_{t+1}\right] \tag{34}$$

where, for any variable X, $\widehat{X}_t = \log\left(\frac{X_t}{X}\right)$ and X is the steady state value; $x^* = \frac{\Phi}{\gamma+\eta}$, $\Phi = \frac{1}{\varepsilon}$.[16]

7.3 The linear-nonlinear model

The linear-nonlinear model is similar to the LQ model except the objective function. Instead of using a quadratic approximation of the objective, I use the exact fully-nonlinear objective:

$$V_t = \frac{C_t^{1-\gamma}}{1-\gamma} - \chi\frac{N_t^{1+\eta}}{1+\eta} + E_t\sum_{j=1}^{\infty}\left[\beta^j\left(\prod_{k=0}^{j-1}\beta_{t+k}\right)\left(\log C_{t+j} - \chi\frac{N_{t+j}^{1+\eta}}{1+\eta}\right)\right] \tag{35}$$

Or in the form of a Bellman equation:

$$V_t = \frac{C_t^{1-\gamma}}{1-\gamma} - \chi\frac{N_t^{1+\eta}}{1+\eta} + \beta\beta_t E_t\left[V_{t+1}\right] \tag{36}$$

[16]Note that the common inflation term $\widehat{\Pi}_t$ does not appear in the quadratic approximation of the objective function due to the fact that I use the Rotemberg price adjustment together with the symmetric equilibrium assumption. With this assumption, the variance of individual prices across firms, and, as a result, the inflation term disappear.

where C_t and N_t are computed as follows:

$$C_t = \exp\left(\widehat{C}_t\right) C \tag{37}$$

$$N_t = \exp\left(\widehat{N}_t\right) C \tag{38}$$

7.4 Steady state values

$$\Pi = 1 + \pi^* \tag{39}$$

$$R = \Pi/\beta \tag{40}$$

$$W = \frac{\varepsilon - 1}{\varepsilon} + \frac{(1-\beta)\,\psi\,(1-\theta)\,\pi\Pi}{\varepsilon} \tag{41}$$

$$N = \left(\frac{W}{\chi\left(1 - \frac{\psi}{2}\left(\pi^* - \theta\pi^*\right)^2\right)^{\gamma}}\right)^{\frac{1}{\gamma+\eta}} \tag{42}$$

$$Y = N \tag{43}$$

$$C = \left(1 - \frac{\psi}{2}\left(\pi^* - \theta\pi^*\right)^2\right) Y \tag{44}$$

$$V = \left(\frac{1}{1-\beta}\right)\left(\frac{C^{\gamma}}{1-\gamma} - \chi\frac{N^{1+\eta}}{1+\eta}\right) \tag{45}$$