Optimal Discretionary Monetary Policy in a Micro-Founded Model with a Zero Lower Bound on Nominal Interest Rate

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Abstract

This paper investigates optimal discretionary monetary policy under the zero lower bound on the nominal interest rate (ZLB) in the case of a distorted steady state due to monopoly and taxation. Solving a fully nonlinear micro-founded (FNL) model using a global method, I find that the central bank in a more distorted economy would cut the interest rate less aggressively under a particular adverse demand shock. This occurs because inflation and nominal interest rates are higher on average, making the ZLB less likely to bind and causing the economy to escape from the ZLB sooner. However, the social planner would choose the optimal inflation rate of approximately zero. The result emerges because the unconditional benefit of avoiding the ZLB is not big enough to offset the cost of higher relative price dispersion when inflation is significantly positive. In addition, I show that the conventional linear-quadratic (LQ) method is inaccurate in the case of a sufficiently distorted steady state.

\textit{JEL classification:} C61, E31, E32, E52.

\textit{Keywords:} optimal discretionary monetary policy, ZLB, distorted steady state, optimal inflation rate, Calvo price adjustments, nonlinear method

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1. Introduction

An extensive literature of optimal monetary policy under the ZLB has studied the case of non-distorted steady state, where the overall economic distortion due to monopoly and taxation is assumed to be zero. In this case, there exists government subsidy to fully offset monopolistic distortion so that the steady state output is not distorted from its socially efficient level. Hence, we can simplify a fully nonlinear micro-founded problem of optimal discretionary monetary policy using the LQ approach developed by Woodford (2001, 2003) and can avoid computational difficulty.

This paper aims at filling the hole in the ZLB literature by investigating optimal discretionary monetary policy under the ZLB in the case of a distorted steady state due to positive overall economic distortion. To this end, I solve a FNL micro-founded model using a global method. Also, I use the LQ method to simplify the FNL model, which I solve using the same method. Then I provide a comparison between the FNL and LQ models.

Studying the case of a distorted steady state brings the ZLB literature in line with the reality. McGrattan (1994) reports that labor income taxes range from 10 – 40%, while Diewert and Fox (2008) estimate that monopolistic markups in some main industries range from 11 – 44%. As a result, the overall economic distortion ranges from 20 – 60%. This information might influence private expectations, which, in turn, would affect optimal policy before, during and after the ZLB period.  

Solving FNL models also helps us to answer the question stated in Adam and

\[ \text{Woodford (2001, 2003)} \]

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to what extent the full nonlinearity affects the optimal monetary policy under discretion in the presence of the ZLB? In addition, we can study the role of relative price dispersion as an endogenous state variable, which is eliminated in the LQ framework due to the linear approximation.

I obtain four sets of main findings. First, under a particular adverse shock driving the economy near the ZLB, the central bank in an economy with a larger economic distortion would cut the interest rate less aggressively. The intuition is simply that, in this economy, inflation and nominal interest rate are higher on average. When the nominal interest rate is near the ZLB, a particular adverse demand shock must have occurred. Given the mean-reverting nature of shocks, the conditional probability that another adverse shock occurs and pushes the economy into the liquidity trap with binding ZLB is very small. Further, even when this shock occurs and the ZLB binds, the output losses and reduction in inflation are smaller than they would be in an economy with a smaller economic distortion. Therefore, downward pressure on the conditional expected inflation is smaller and the central bank cuts the nominal interest rate less aggressively.

Second, with a larger overall economic distortion, inflation and interest rates are higher on average, resulting in smaller probability of reaching the ZLB. However, the social planner would choose the optimal inflation rate of approximately zero, corresponding to very small overall economic distortion. This occurs because the unconditional benefit from avoiding the ZLB is not big enough to offset the cost of higher relative price dispersion when inflation is high. In sum, the unconditional expected welfare is maximal when the average long run inflation is around zero.

Alvarez et al. (2011) find that relative price variance is significantly positive when inflation is high, while Zandweghe and Wolman (2010) shows that initial relative price dispersion could affect monetary policy. So studying the role of relative price dispersion is interesting.
Third, when the initial relative price distortion is greater than the steady state value, the central bank tends to pursue higher nominal interest rates, making the ZLB less likely to bind. The intuition is that the relative price dispersion is an inefficiency wedge, when it is high the central bank would like to reduce it by tightening the monetary policy and, as a result, lowering the front-loading behavior by firms in setting their prices, leading to a smaller current relative price dispersion. This result is interesting and can not be found using the LQ method because the change in relative price dispersion is always zero.

Finally, the FNL model and the LQ model produce different results if there is a particular adverse shock that makes the ZLB binding. The reason is that, when the ZLB binds, the central bank cannot stabilize output and the price level, making the relative price dispersion to stay far away from the steady state. While the impact of the relative price dispersion as an endogenous state variable in the FNL model is significant, it is always zero and has no role in the LQ model due to the first order approximation. However, the difference between the FNL and LQ model is not significant in the case of non-distorted steady state.

When the overall economic distortion is large, the two methods produce very different results, especially when the ZLB binds. In this case the approximated inflation and interest rate in the LQ model are substantially smaller than the true values, derived using the FNL model. Consequently, given the ZLB binds in both models, the output losses in the FNL model are significantly smaller than those in the LQ model. In addition, the interest rate cut in the FNL model is less aggressive under a shock driving the interest rate near the ZLB.

The related literature on optimal monetary policy under the ZLB has been inspired by seminal work by [Krugman 1998], which extensively discusses causes and consequences of the ZLB in a series of simple two-period perfect-foresight models.
Since then, extensive research related to the ZLB has been implemented, including Eggertsson and Woodford (2003), Jung et al. (2005), Adam and Billi (2006), Nakov (2008), Levin et al. (2010), Bodenstein et al. (2010), Eggertsson and Krugman (2010), and Werning (2011). The common feature of these papers is that they focus on the case of non-distorted steady state and use the LQ method.

The papers closest to mine are Adam and Billi (2007), and Anderson et al. (2010). Unlike the previous ZLB literature, Adam and Billi (2007) use a global method to solve an optimal monetary policy problem under discretion that allows for an occasionally-binding ZLB. However, because they use the LQ approach, the only nonlinearity in their paper is the ZLB. This paper extends Adam and Billi (2007) by considering a fully nonlinear model. In addition, this paper studies the implications of positive overall economic distortion on discretionary policy and optimal inflation rate in the presence of the ZLB.

Anderson et al. (2010) investigate the size of inflationary bias under discretion in the presence of overall economic distortion using nonlinear methods. However, in their model, the nominal interest rate can be adjusted freely because the ZLB is not allowed. So the average long-run inflation is the same as the deterministic steady state inflation. This paper extends their work by considering the ZLB, a very important constraint faced by policymakers.

There are three recent working papers studying the ZLB using fully nonlinear methods. Nakata (2011) studies optimal fiscal and monetary policy in a nonlinear sticky price model of the Rotemberg-type instead of the Calvo-type as in my model. I choose to use the Calvo-type price adjustments so that I can examine relative price dispersion as an endogenous variable and compare my results directly to the results in the previous literature. I study the role of economic distortion, while he focuses on fiscal policy. Fernandez-Villaverde et al. (2012) study the ZLB in a fully nonlinear
model using the collocation method associated with Smolyak nodes. Judd et al. (2011) solve a fully nonlinear New Keynesian model with the ZLB using a cluster-grid algorithm. The monetary policy in these two papers is a Taylor rule. They do not consider optimal discretionary monetary policy.

The remainder of this paper is organized as follows. Section 2 presents the structure of the economy. Section 3 describes the discretionary monetary policy problem faced by a central bank and explains briefly solution methods. In Section 4, we calibrate key parameters and report main results. Section 5 concludes.

2. Model

The economic structure in this paper presents two key New Keynesian features, such as in Rotemberg and Woodford (1997) and Yun (1996). Particularly, intermediate goods producers are monopolistic competitors. In addition, they reset their prices infrequently à la Calvo (1983).

2.1. Household

The representative household maximizes his total expected discounted flow utilities:

$$\max E_t \left\{ \left( \frac{C_t^{1-\gamma}}{1-\gamma} - \chi \frac{N_t^{1+\eta}}{1+\eta} \right) + \sum_{j=1}^{\infty} \left\{ \prod_{k=0}^{j-1} \beta_{t+k} \right\} \left( \frac{C_t^{1-\gamma}}{1-\gamma} - \chi \frac{N_t^{1+\eta}}{1+\eta} \right) \right\}$$

subject to the budget constraint:

$$C_t + B_t = (1 - \tau_w) w_t N_t + B_{t-1} \left( \frac{1 + \hat{i}_{t-1}}{1 + \pi_t} \right) + \int_0^1 D_t(i) di + T_t$$

(1)
where $C,N$ are composite consumption and total labor; $B,D,T$ denote real bond, dividend and lump sum transfer; $i,\pi$ are net nominal interest rate and inflation, respectively; $w$ is real wage; $\tau_w$ is labor income tax; $\gamma, \eta, \chi$ are risk aversion, inverse wage elasticity of labor, and steady state labor determining parameters; $\beta$ is the stochastic subjective discount factor or preference shock that follows an AR(1) process with a steady state value $\beta$:

$$\ln (\beta_{t+1}) = (1 - \rho_\beta) \ln (\beta) + \rho_\beta \ln (\beta_t) + \varepsilon_{\beta,t}, \text{ where } \beta_t \text{ is given}. \quad (2)$$

The optimal choices of the household must satisfy the following conditions:

$$E_t \left[ \beta_t \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \left( \frac{1 + i_t}{1 + \pi_{t+1}} \right) \right] = 1 \quad (3)$$

$$\frac{\chi N_t^\eta}{C_t^{-\gamma}} = (1 - \tau_w) w_t \quad (4)$$

The first condition shows the marginal intertemporal trade-off between today’s and tomorrow’s consumption. The second condition is the marginal trade-off between working and consuming.

The stochastic preference is a reduced form of more realistic forces that can drive the nominal interest rate to the ZLB. From the Euler equation, an increase in discount factor causes the nominal interest rate to fall given private expectations and households’ desire to smooth their consumption.\(^3\)

\(^3\)The conventional technology shock is not able to cause the ZLB to bind realistically. The reason is that we need a very big positive technology shock to generate massive savings that can drive the nominal interest rate to the ZLB. We did not observe this type of shock before the onset of the last crisis, see Amano and Shukayev (2012). Guerrieri and Lorenzoni (2011) model debt limit and household heterogeneity in labor produc-
To produce the composite consumption goods, $C_t$, the household buys and aggregates variety of intermediate goods using a CES technology. His cost-minimization problem is given below.

$$\min \int_0^1 P_t(i) C_t(i) \, di \quad \text{s.t.} \quad C_t = \left( \int_0^1 C_t(i)^{\frac{1}{1-\epsilon}} \, di \right)^{1-\epsilon} \tag{5}$$

where $C_t(i)$ is the amount of intermediate goods $i \in [0, 1]$ and $\epsilon$ is the elasticity of substitution between intermediate goods.

The optimal condition gives rise to the demand for the intermediate goods $i$, $C_t(i)$, and the aggregate price level, $P_t$, below:

$$C_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\epsilon} C_t \tag{6}$$

$$P_t = \left( \int P_t(i)^{1-\epsilon} \, di \right)^{\frac{1}{1-\epsilon}} \tag{7}$$

Eggertsson and Krugman (2010) also model household debt limit and deleveraging as a key factor to drive the nominal interest rate to the ZLB. In their model, an initial shock to the debt limit causes borrowers to deleverage by cutting back their consumption, resulting in a decrease in the price level. This deflation puts more pressure on the real debt the borrowers have to pay back now, leading to further deleveraging and a sharper decline in the nominal interest rate.

Ngo (2013) extends Eggertsson and Krugman (2010) by endogenizing the debt limit. He studies the interaction between the ZLB and the endogenous debt limit in explaining the collapse of the housing market and the Great Recession. Hall (2011) models excessive capital stock and a sharp decline in capital utilization as the reason for the nominal interest rate to be pinned at the ZLB. Curdia and Woodford (2009) model a shock to the wedge between deposit and lending rates as a driving force.
2.2. Intermediate goods producers

There is a mass one of intermediate goods producers that are monopolistic competitors. Each period a firm keeps its previous price with probability $\theta$ and resets its price with probability $(1 - \theta)$.

Given its price $P_t(i)$ and demand $Y_t(i)$, the firm $i$ chooses labor that

$$\min \{ w_t N_t(i) \} \quad s.t. \quad Y_t(i) = N_t(i) \quad (8)$$

Let $\varphi_t(i)$ be the Lagrange multiplier with respect to the production. The first order condition gives the same marginal cost to all firms, $\varphi_t$:

$$\varphi_t = \varphi_t(i) = w_t \quad (9)$$

Whenever a firm has a chance to reset its price, it chooses the new price to solve:

$$\max_{P_t(i)} E_t \left\{ \left[ \frac{P_t(i)}{P_t} - \varphi_t \right] Y_t(i) + \sum_{j=1}^{\infty} \left\{ \theta^j \left( \prod_{k=0}^{j-1} \beta_t \right) \left( \frac{C_t}{C_t} \right)^{-\gamma} \left[ \frac{P_t(i)}{P_{t+j}} - \varphi_{t+j} \right] Y_{t+j}(i) \right\} \right\}$$

subject to its demand in equation (6).

The optimal relative price, $P_t^*(i)/P_t$, is the same for all firms who are able to reset their prices today:

$$\frac{P_t^*(i)}{P_t} = p_t^* = \frac{\left( \frac{\varepsilon}{\varepsilon-1} \right) E_t \left\{ C_t^{-\gamma} Y_t \varphi_t + \sum_{j=1}^{\infty} \left\{ \theta^j \left( \prod_{k=0}^{j-1} \beta_t \right) C_t^{-\gamma} \left( \frac{P_{t+j}}{P_t} \right)^{\varepsilon-1} Y_{t+j} \varphi_{t+j} \right\} \right\}}{E_t \left\{ C_t^{-\gamma} Y_t + \sum_{j=1}^{\infty} \left\{ \theta^j \left( \prod_{k=0}^{j-1} \beta_t \right) C_t^{-\gamma} \left( \frac{P_{t+j}}{P_t} \right)^{\varepsilon-1} Y_{t+j} \right\} \right\}} \quad (11)$$
With some manipulation, we can rewrite the optimal pricing rule as below:

\[ p_t^* = \frac{S_t}{F_t} \]  

(12)

where \( S_t, F_t \) are written in the following recursive forms:

\[ S_t = \left( \frac{\varepsilon}{\varepsilon - 1} \right) C^{-\gamma}_t Y_t \varphi_t + \theta E_t [\beta_t \Pi_{t+1}^{\varepsilon} S_{t+1}] \]  

(13)

\[ F_t = C^{-\gamma}_t Y_t + \theta E_t [\beta_t \Pi_{t+1}^{\varepsilon} F_{t+1}] \]  

(14)

and \( \Pi = (1 + \pi) \) is gross inflation.

Combining (13) with (4) and (9), we obtain:

\[ S_t = \frac{\chi C_t N_t^\eta}{(1 - \Phi)} + \theta E_t [\beta_t \Pi_{t+1}^{\varepsilon} S_{t+1}] \]  

(15)

where

\[ \Phi = 1 - (1 - \tau_w) \cdot (1 - \varepsilon^{-1}) \]  

(16)

and \( \Phi \) is called overall economic distortion. I will discuss this metric in a section below.

2.3. Aggregate conditions

Aggregate output satisfies:

\[ Y_t = \frac{N_t}{\Delta_t} \]  

(17)

where \( \Delta_t \) is called the relative price dispersion and is defined as:

\[ \Delta_t = \int \left( \frac{P_t (i)}{\bar{P}_t} \right)^{-\varepsilon} di \]  

(18)
or in a recursive form:

$$\Delta_t = \theta \Pi_t^{\varepsilon} \Delta_{t-1} + (1 - \theta) (p_t^*)^{-\varepsilon}$$

(19)

I write the price level $p_t^*$ in a recursive form and divide both sides by $P_t$ to obtain the optimal relative price:

$$p_t^* = \left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \theta} \right)^{1/\varepsilon}$$

(20)

Plugging this optimal relative price in the relative price dispersion equation (19) we obtain:

$$\Delta_t = (1 - \theta) \left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \theta} \right)^{\varepsilon^{-1}} + \theta \Pi_t^{\varepsilon} \Delta_{t-1}$$

(21)

2.4. Overall economic distortion

In this section, I discuss the overall economic distortion, which is defined as in equation (16). To understand more about the meaning of this notation, let us consider an economy with flexible price. In this economy, the marginal cost, $\varphi$, equals the inverse of markup (or $(1 - \varepsilon^{-1})$). From equation (4), (9), and (17) we compute the equilibrium flexible-price output ($Y_t^f$) and the equilibrium efficient output ($Y_t^*$) as follows:

$$Y_t^f = N^* \cdot (1 - \Phi)^{1/(\eta + \gamma)}$$

(22)

$$Y_t^* = N^*$$

(23)

where $N^*$ is the long-run efficient output/labor. The percentage deviation of the flexible-price output from the efficient output equals:

$$\left( \frac{Y_t^f - Y_t^*}{Y_t^*} \right) \cdot 100 \simeq - \frac{1}{\eta + \gamma} \cdot \Phi \cdot 100$$
The larger the overall economic distortion, the smaller the flexible-price output relative to the efficient output. It is important to note that while the overall economic distortion is zero, it does not mean there is not any type of economic distortions. Instead, it means that we can attain the efficient output level by designating labor income subsidy to fully offset the monopoly power, given no price stickiness.

It is also important to emphasize that when the overall economic distortion is large or the inverse labor elasticity and risk aversion are small, the flexible price output is far below the efficient output level. Under discretion, the central bank tends to create positive inflation to try to attain the efficient output. In equilibrium, the greater the overall economic distortion, the smaller the flexible-price output relative to the efficient output, the greater the inflation the central bank tends to create.

3. Optimal discretionary policy problem under the ZLB

The central bank takes the expectations of economic agents as given and maximizes the representative household’s discounted utility subject to the optimality conditions from market participants, the aggregate conditions, the law of motion for the state variables, and the explicit ZLB on the nominal interest rate. Under discretion, the economic agents know that the central bank is going to re-optimize every period and they incorporate this information in forming their expectations. As a result, unlike under commitment, under discretion the central bank does not have power in manipulating the private expectations.\footnote{The central bank can manipulate private expectations under commitment by pre-committing to the path of current and future policies. However, the time-inconsistency issue arises and commitment is not credible.}
The problem can be stated in the form of a Bellman equation:

\[
V(\Delta_{t-1}, \beta_t) = \max \left\{ \frac{C_{t}^{1-\gamma}}{1-\gamma} - \chi \frac{N_{t}^{1+\eta}}{1+\eta} + \beta_t E_t V(\Delta_t, \beta_{t+1}) \right\}
\]  

subject to

(i) Households’ and firms’ optimality conditions, and aggregate conditions.

(ii) Law of motion for state variables.

(iii) ZLB on the nominal interest rate \((i_t \geq 0)\).

(iv) No commitment to future policy that is made in the past.\(^5\)

The solution of the above nonlinear system is called Markovian invariant policy function of the state, \(s_t = (\Delta_{t-1}, \beta_t)\), where \(\Delta_{t-1}\) is an endogenous state and \(\beta_t\) is an exogenous one. In the paper, I solve the above FNL model using a global method called collocation method. First, I use equidistant collocation nodes to solve the model and find out policy function, based on which I investigate potential kinks. Then I redistribute the nodes by clustering them around these potential kinks and resolve the model. I employ the time-iteration method. At each collocation node, I solve a complementarity problem using the Newton method and semi-smooth root finding formulations as described in Miranda and Fackler (2002). I also provide an "analytical" Jacobian matrix computed from the approximating functions.\(^6\) Moreover, I write my code using a parallel computing method that allows us to split up a large number of collocation nodes into smaller groups that then are assigned to different processors to solve simultaneously. All these computational characteristics help to significantly increase the rate of convergence and make the solution method

\(^5\)See Appendix A for how to write down the problem in detail.\(^6\)See Appendix C for the "analytical" Jacobian matrix.
very reliable.

I also use the LQ approach, as described in Woodford (2003), to simplify the FNL model which I then solve using the same method. Specifically, according to the LQ approach, the central bank’s objective function is quadratically approximated and all the constraints and law of motion are (log)linearly approximated around the steady state values associated with zero inflation. The endogenous variables in the LQ framework are defined as below:

\[
\hat{\pi}_t = \log (1 + \pi_t) - \log(1) \tag{25}
\]

\[
\hat{i}_t = \log (1 + i_t) - \log(1/\beta) \tag{26}
\]

\[
x_t = \left( \log(Y_t) - \log(\bar{Y}) \right) - \left( \log(Y^f_t) - \log(\bar{Y}^f) \right) \tag{27}
\]

where $\beta$ is the steady state discount factor; $\bar{Y}$ and $\bar{Y}^f$ are the steady state sticky-price and flexible-price outputs when the overall economic distortion presents; $x$ is output gap. First, we solve for the policy function in the LQ framework, including $\hat{i}_t, \hat{\pi}_t, x_t$. Then, we back out the policy function for $i_t, \pi_t, Y_t$ using equations (25) − (27), which I call LQ results.

See Appendix B for how we solve the model in detail, including the error reported from checking the solution.

See Appendix D for the simplified LQ model. Note that (i) the LQ model with the ZLB is also a nonlinear model and I have to use the global method to solve it; (ii) the LQ approach is actually not applicable when the overall economic distortion, $\Phi$, is large.
4. Results

4.1. Parameter calibration

I calibrate the steady state quarterly time discount factor, $\bar{\beta}$, to be 0.993, corresponding to a real interest rate of 2.8% per year. The relative risk aversion ($\gamma$) is 4, as in Nakov (2008), and the inverse elasticity of labor with respect to wage ($\eta$) is 1. The monopoly power parameter ($\varepsilon$) is calibrated to be 10, corresponding to a 11% markup that is in the range found by Diewert and Fox (2008). The probability that a firm keeps its price unchanged each quarter, $\theta$, is chosen to be 0.75 so that firms keep their prices for 4 quarters on average. This value is commonly used in the literature, such as Anderson et al. (2010).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Values</th>
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<tbody>
<tr>
<td>$\bar{\beta}$</td>
<td>Quarterly discount factor</td>
<td>0.993</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Constant relative risk aversion</td>
<td>4</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Inverse elasticity of labor with respect to real wage</td>
<td>1</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Monopoly power</td>
<td>10</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Probability that a firm keeps its price unchanged each quarter</td>
<td>0.75</td>
</tr>
<tr>
<td>$\sigma_{\beta}$</td>
<td>Standard deviation of preference preference shocks (percent)</td>
<td>0.4</td>
</tr>
<tr>
<td>$\rho_{\beta}$</td>
<td>AR-coefficient of preference shocks</td>
<td>0.8</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Overall economic distortion</td>
<td>0; 0.20</td>
</tr>
<tr>
<td>$\chi$</td>
<td>Parameter associated with the disutility of labor</td>
<td>1</td>
</tr>
</tbody>
</table>

I calibrate the persistence of the preference shock to be 0.8 that is consistent with the persistence of the natural rate of interest rate as in Adam and Billi (2007). The hard part is how to calibrate the variance of the preference shock. In the paper, I
calibrate this parameter to be 0.42% per quarter that enables the model to generate the unconditional probability of hitting the ZLB of around 6%. This value is a little small compared with the fact that we have been at the ZLB since December 2008 and that we are projected to be at the bound until mid 2015. However this value is still the upper value of the range 5% – 6% found in the empirical studies before the last financial crisis, as in Fernandez-Villaverde et al. (2012).

The overall economic distortion, Φ, is calibrated to be either 0 or 0.20. The first value corresponds to the well-known non-distorted steady state. The second value corresponds to the case where labor income tax is set to be 11%. Although, the tax rate is conservative, it is still in the range found by Diewert and Fox (2008). As I show below, a higher value of Φ only makes the LQ model more inaccurate.

4.2. Steady state

The steady state values depend on the overall economic distortion (Φ). With a labor income subsidy designed to fully offset the monopolistic distortion, the overall economic distortion is zero. In this case, the steady state inflation and gross interest rate are 0 and $1/\beta$ respectively. However, in the case of positive overall economic distortion, it is difficult to compute the steady state values.

Figure 1 shows optimal gross inflation and relative price dispersion as a function of initial relative price dispersion, given the steady state preference of $\bar{\beta}$. The steady state relative price dispersion is the value that equals the initial relative price dispersion. In this example, they are 1.0029 (or about 1.2% annually). Using this value, we compute the steady state gross inflation to be 1.006 (or 2.4% per year). See Appendix for the formula that can be used to compute the steady state inflation using the LQ method.

To illustrate the impact of nonlinearity and overall economic distortion, I compute
the deterministic steady state inflation in both LQ and FNL models with respect to different values of the overall economic distortion ($\Phi$). The results are presented in Figure 2 and are similar to Anderson et al. (2010).

Two interesting features in Figure 2 are worth being addressed. First, there is a positive relationship between the steady state inflation and the size of overall economic distortion in both FNL and LQ models. Intuitively, the larger the size of

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Note that the monopolistic distortion is always 0.1. For each value of overall economic distortion ($\Phi$), we can compute a corresponding value of income tax using equation (16). For example, if $\Phi = 0$, $\tau_w = -11\%$; if $\Phi = 0.2$, $\tau_w = 11\%$.
the overall distortion, the higher the marginal benefit of inflation because a higher inflation rate can help to lower the real markup, stimulating output and employment. However, this does not come at no cost. In fact, a higher inflation rate induces a firm to set a higher price when it has a chance to do so because the firm knows that it may not be able to adjust its price in the future and that a higher inflation rate will erode its relative price and profit. This front-loading in setting price causes the dispersion in relative prices to increase and lower the aggregate output, as in equation (17).

Second and more importantly, Figure 2 shows that the steady state interest rate in the true (FNL) model is a convex function with respect to the size of overall economic distortion.
distortion. However, the steady state interest rate in the LQ model are only the first order approximation of the true value around $\Phi = 0$. Due to the convexity of the true function, the LQ model always underestimates the true steady state value.\textsuperscript{10} When the size of the overall economic distortion increases, the underestimation increases at an increasing rate.

Surprisingly, it is not difficult to prove that, under commitment, the steady state inflation rate in both LQ and FNL models is zero regardless of the size of overall economic distortion.\textsuperscript{11} Therefore the steady state interest rate is always equal to the steady state real interest rate, which is 2.8% annually.

With inflation and interest rates being smaller than the true ones, the ZLB is more likely to be reached in the LQ model. Therefore, given preference shocks that cause the ZLB to bind in both models, the LQ method generates more sizeable output losses than does the FNL method. We will see this more clearly in the next section.

4.3. Optimal output, inflation, and interest rate policy

When there is a positive preference shock, households value their future consumption more. In other words, they are more patient so they tend to save more and consume less, putting downward pressure on output and the price level. To restore consumption and output, we need a lower real rate. If the central bank was not restrained by the ZLB, he could adjust the nominal interest rate so that the actual real interest rate is the same as the natural real rate. However, because the ZLB is allowed, a big positive preference shock causes the ZLB to bind. As a result,

\textsuperscript{10}While the degree of convexity depends negatively on the curvature of the labor supply ($\eta$) and the risk aversion parameter ($\gamma$), it is positively related to the price stickiness.
\textsuperscript{11}See Schmitt-Grohe and Uribe (2010) for the proof.
the actual real rate will be larger than the natural real rate because the nominal interest rate cannot be negative, resulting in a sizable output loss. For comparison, I experiment with different levels of overall economic distortion and relative price dispersion.

4.3.1. The case of a non-distorted steady state

In this efficient economy, there exists a labor income subsidy designed to fully offset the monopoly power so the overall economic distortion ($\Phi$) is zero. Figure 3 shows the policy function at each value of the preference shock given the initial relative price dispersion at the steady state.

Figure 3 shows the equilibrium responses from the FNL model, given the initial relative price dispersion at the steady state. The solid blue lines represent the results from the model with ZLB, while the dashed green lines represent those from the model without ZLB. The results have the same characteristics as those in Adam and Billi (2007) and Nakov (2008). First, in the absence of the ZLB, the central bank can achieve the target efficient output and price stabilization by adjusting the nominal interest rate as much as possible, even to be negative. Second, when the ZLB presents, the central bank cannot stabilize output and inflation under shocks that cause the ZLB binding. Third, the central bank cuts the nominal interest rate more aggressively, especially when the economy is near the ZLB, in the model with the stochastic ZLB than in the model without the ZLB or with perfect foresight binding ZLB. The aggressiveness occurs due to the risk of falling into the liquidity trap associated with deflation that makes the central bank to cut the interest rate more than it would be without the risk.\footnote{See Adam and Billi (2007) and Nakov (2008) for more detailed explanation.}
Adam and Billi (2007) ask whether a fully-nonlinear model might generate policy function different from theirs. By solving both the FNL and LQ models, I am able to answer the question.\textsuperscript{13} The red dot-dashed lines in Figure 3 present the policy function using the LQ framework. Without a particular positive shock, the optimal policy is very similar in the two models. The finding is robust to the parameters and

\footnotesize\textsuperscript{14}Fernandez-Villaverde et al. (2012) claim that results from a fully nonlinear model are very different from those in the LQ model. However, they model monetary policy using a Taylor rule with an inflation target of 2\% instead of zero inflation target as in this part. Judd et al. (2011) compare the results from their nonlinear method with those from the perturbation method, not with the LQ method.

Figure 3: Optimal policy in the economy with a non-distorted steady state ($\Phi = 0$). Gross initial relative price dispersion is at the steady state ($\Delta_{t-1} = \Delta^{SS} = 1$).
nature of shocks. The reason is that, when $\Phi = 0$, the steady state inflation and interest rate are the same in both models regardless of the parameters and nature of shocks. Also, the relative price dispersion is zero in the two models.

However, when a particular positive shock occurs and the ZLB binds, the central bank cannot stabilize the price level, so the current relative price dispersion increases from the steady state and starts playing its role, as a negative technology shock, in the FNL model. In this case, the FNL model generates more output loss and more decline in the price level than does the LQ model, which keeps the relative price dispersion constant regardless of the state of the economy, as in Panel D of Figure 3. However, the difference between the FNL and LQ models are not significant.

To investigate the role of initial relative price dispersion under the ZLB, I plot the optimal policy using different values of initial relative price dispersion, as in Figure 4. The solid blue lines show the policy function when the initial relative price dispersion is 0% annually, while the dash-dotted red lines and dashed green lines show the policy function when the initial relative price dispersion is at 3.5% and 10% respectively.

Note that the initial relative price dispersion ($\Delta_{t-1}$) can be very high due to a change in the tax regime, although the responses of current relative price dispersion ($\Delta_t$) are relatively small under preference shocks. For example, when the labor income tax changes from $\tau_w = 16.67\%$ initially to $\tau_w = -11.0\%$ as in the case of non-distorted steady state, the initial relative price dispersion is 5.2% per year. The larger the tax change, the greater the intial relative price dispersion.

As shown in Figure 4, when the initial relative price dispersion is 3.5%, the nominal interest rate is about 0.7% higher than it would otherwise if the initial relative price dispersion is zero. The current relative price dispersion is about 2.5%, which is 1.0% lower than the initial value. The output loss is about 1.0%, due to
high dispersion of relative prices. The economy experiences deflation.

Intuitively, when the initial relative price dispersion is large, the inefficiency wedge is high, and the central bank would implement highly contractionary monetary policy by pursuing higher nominal interest rates on average than it would otherwise. By doing so, the central bank can lower the front-loading price setting behavior of firms and, as a result, lower the current relative price dispersion. In this case, the monetary policy is so contractionary that it creates output losses and disinflation (or deflation in this case). Interestingly, due to high nominal interest rates on average the ZLB is less likely to bind. For example, in the case of 3.5% initial relative price dispersion,
the ZLB binds only when there is a shock with a magnitude of at least 2.5% occurring, compared with 2% in the zero initial relative price dispersion.

4.3.2. The case of a distorted steady state

As explained in the calibration section, in this case, the overall economic distortion (Φ) in this economy is 0.20. This means that at the steady state, the economy produces much less than the efficient output level. With this overall economic distortion, the central bank no longer targets zero inflation. The deterministic steady state inflation is about 2.4% that is associated with the steady state interest rate and price dispersion of 5.2% and 1.2%, respectively. Figure 5 shows the optimal policy where the relative price dispersion is set at the steady state of 1.0029 (or 1.2% annually).

The solid blue lines show the policy function in the FNL model. We can easily see that on average the central bank pursues higher inflation and nominal interest rate in the FNL model than in the LQ model. Without a particular positive preference shock, the central bank implements the inflation rate and interest rate of around 2.4% and 5.2% respectively. The higher the average inflation and interest rates, the less likely the ZLB will bind.

In the first case of zero overall economic distortion, the steady state inflation and interest rate are 0 and 2.8% respectively. A positive preference shock with a magnitude of 1.2 standard deviations, which reduces the natural real rate by 2% annually, can make the ZLB to bind in the first case. However, in this case of a distorted steady state, with 2.4% inflation target and 5.2% steady state interest rate, it requires a much more severe shock to drive the economy to the ZLB - about 3 standard deviations (or reducing the natural real rate by 5% annually).

More importantly, when the economy is near the ZLB, the central bank in the economy with zero overall economic distortion cuts interest rate more aggressively
than in the economy with positive distortion. The intuition is that, in the case of a large overall economic distortion, the central bank pursues a positive inflation target. Hence, the nominal interest rate is high on average.

When the economy to be near the ZLB, a particular adverse preference shock must have occurred. Because the preference process is mean-reverting, it is rather unlikely that another adverse shock will happen and push the economy into the liquidity trap with output loss and low inflation. As a result, the downward pressure on the conditional expected inflation is very small, generating small pressure on
lowering further the nominal interest rate. Therefore, interest rate cut is not as big as it would be in an economy with a smaller overall economic distortion.

The literature studying the case of non-distorted steady state, including Adam and Billi (2007), indicates that even when the economy escapes from a liquidity trap (or when output is higher than the steady state value) the central bank still keeps the nominal interest rate at zero for some time. We can see this conclusion in the case of non-distorted steady state of this paper too. However, with conservative overall economic distortion as in this section, the central bank should not keep the nominal interest rate for some time when the economy escapes from the liquidity trap.

Unlike the case of a non-distorted steady state, in this case the ZLB and liquidity trap are not necessarily associated with deflation. The reason is that the expected inflation is high in this case. Even if an adverse demand shock occurs and pushes the economy into the liquidity trap with output loss and binding ZLB, downward pressure on the price level may not be big enough to offset the high expected inflation. So, the actual inflation is positive. For example, at the preference of 6% higher than the steady state value (or the natural real rate is 6% lower than its steady state rate), output loss is 0.3% that is associated with 2.3% inflation rate. The results with positive overall economic distortion is consistent with what we have observed since the last recession where the federal funds rate (FFR) is technically zero and inflation is moderately positive.

The policy function from the LQ framework is presented by the dash-dotted red lines in Figure 5. They are less accurate compared with the true policy in the FNL model. Specifically, in the LQ model, it requires a shock with a magnitude of 2.3 standard deviations (or 4% per year) to drive the economy to the ZLB, while it requires a shock with a magnitude of at least 3 standard deviations (or 5.2% per year) to make the ZLB reaching in the FNL model. In addition, given that the ZLB
is binding in both models, the output loss and inflation decline are much larger in the LQ model. Particularly, when the time discount factor is 8% higher than its steady state value (or the natural real rate is 8% smaller than its steady state), the bounds are binding in both models. The output falls by 2.2% associated with an inflation rate decline of 1.7% in the LQ model, compared to 1.25% and 0.4% in the FNL model.

This occurs due to the fact that the central bank in the LQ approach pursues lower inflation and nominal interest rate on average because of the inaccuracy of the LQ approach. Intuitively, the inaccuracy of the LQ model comes from the fact that the LQ method eliminates the endogenous state variable called relative price dispersion. As shown in Panel D of Figure 5, in the LQ framework, the relative price dispersion is always 0% annually, while it is 1.2% in the FNL model. And we know that higher relative price dispersion is associated with higher inflation, as in equation (21). So an adverse shock that causes the ZLB to bind in the FNL model to bind must generate more slackness in the LQ model. As a result, the output loss and decline in inflation are greater in the LQ model than in the FNL model.

Figure 5 also shows that when the economy is near the ZLB, the interest rate is cut more aggressively in the LQ model than in the FNL model. The intuition is the same as above. With a larger overall economic distortion, inflation and nominal interest rate in the LQ model are smaller than those in the true model, which is the FNL model. Therefore, the ZLB in the LQ model is more likely to bind and the economy is more likely to be pushed in the liquidity trap with lower inflation. When the economy is near the ZLB, downward pressure on the conditional expected inflation in the LQ model is larger than in the FNL model, resulting in a more

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14 The shock is slightly above three standard deviations.
aggressive cut in interest rate.

To provide more detailed comparison of the FNL and the LQ model, it is useful to answer the question: to which extent the equilibrium responses in the case of a distorted steady state are really driven by the presence of the ZLB? From Figure 5, the central bank can obtain the inflation and output targets using both FNL and LQ models in the absence of the ZLB. There are only demand shocks (or preference shocks) in the model so there is no trade-off between output and inflation stabilization. However, the targets based on the LQ method are inaccurate because this method does not capture the full nonlinearity. Specifically, from Panel B of Figure 5, the equilibrium inflation rate is always 2.4% per year in the FNL model and 1.6% in the LQ model. The difference of 0.8% between the two models is solely due to the nonlinearity.

In the presence of the ZLB, the equilibrium responses in the FNL and LQ model are different due to two factors. The first factor is the nonlinearity effect as explained above. The second factor is the presence of the ZLB. To see the second effect, let us examine Panel B of Figure 5 again. Under a shock that increases the preference by 6% per year and the ZLB binds, the equilibrium inflation is about 2.3% in the FNL model and 1% in the LQ model. The total difference is about 1.3%. Because the nonlinearity accounts for approximately 0.8%, the ZLB presence accounts for approximately 0.5%.

To extend the results from Yun (2005), I investigate the role of initial relative price dispersion and report the results in Figure 6. When the initial dispersion is greater than the steady state value of 1.2%, both output gap and inflation fall further than they would if the initial dispersion is kept at the steady state value. The opposite results occur if the initial dispersion is smaller than the steady state value. Again, this happens because the relative price dispersion is positively correlated with the
initial dispersion as we see in Figure 1.

In addition, the relative price dispersion plays the role of endogenous technology in the aggregate production function. The higher the relative price dispersion, the lower the technology and the lower the output. Therefore, the additional output loss (gain) depends on whether the initial relative price dispersion is greater (smaller) than its steady state value. From Figure 5 we also see that the greater the initial relative price dispersion, the higher the nominal interest rate. As a result, the ZLB is less likely to bind.

Figure 6: Optimal policy in the economy with a distorted steady state (Φ = 0.2), with different values of initial relative price dispersion, in the FNL model.
4.4. What is the optimal inflation rate?

Since the late 1990s when Japan fell into the liquidity trap with binding ZLB, economists, such as Krugman (1998), have been debating about whether central banks should target a significantly positive inflation target and what the optimal inflation rate is. The topics become even more important nowadays as the U.S. federal funds rate have been at zero since December 2008 while the US economy has been experiencing the greatest slump after the Great Depression. Blanchard et al. (2010) suggest that policymakers might consider a optimal inflation target of around 4%. The suggestion lies under the argument that, in the presence of the ZLB, significantly positive inflation creates leeway for the central bank to deal with a particular adverse demand shock that would drive the economy into the liquidity trap with binding ZLB.

However, positive inflation does not come at no cost. Higher inflation is always associated with more front-loading behavior of firms when they have a chance to reset their prices. As a result, it is associated with higher relative price dispersion and lower output. This occurs because if the firms know that inflation is high and they cannot adjust their prices flexibly in the future, they will set higher prices today, causing higher relative price dispersion.

In this section, I am going to use the FNL model to answer a very important policy question - what is the optimal inflation target the social planner should pursue by setting the size of overall economic distortion or tax rate accordingly, as in equation (16)? For example, the social planner can choose a 2% inflation target by setting overall economic distortion of 0.18, or 9% income tax. It is reasonable to think of the social planner having two separate decision making bodies. One is the Treasury Department that conducts tax policy and the other is the central bank that conducts monetary policy. The social planner is assumed to be able to choose and commit
to a labor income tax policy knowing that he will implement optional discretionary monetary policy later.

The social planner’s problem boils down to comparing the social welfare for each tax/subsidy policy under discretionary monetary policy, choosing the best tax policy once, then committing to the policy. The setup is very similar to the case of discretionary policy without model misspecification as in [Billi (2011)], where the social planner chooses the inflation goal once and for all at time $t=0$. Absent the ZLB, the social planner should choose a zero inflation goal and, as a result, zero optimal long run inflation. However, in the presence of the ZLB, the social planner should choose positive inflation target as the guard against the incidence of the ZLB.

To find out the optimal inflation rate, first I solve the FNL model with respect to different values of the overall economic distortion. Then, based on the optimal policy, I compute the corresponding inflation target, which is the average long run inflation from a simulation of 300,000 periods. Also, I compute the simulated probability of hitting the ZLB using these 300,000 period. As shown in Figure 7, with the y-axis in the left, the higher the overall economic distortion, the higher the inflation target and the lower the probability of hitting the ZLB. For example, if the overall economic distortion is zero, or the income tax rate is $-11\%$, the average long run inflation is around $-0.02\%$ and the probability of hitting the ZLB is around 6%.

The solid red line in Figure 7, with the y-axis in the right, presents the unconditional welfare relative to the one associated with the non-distorted steady state, as the function of overall economic distortion. To compute this unconditional relative welfare, for each value of overall economic distortion, I first solve for the value function as the function of the initial relative price distortion and preference shock. Then, I take a random sample of 300,000 preference shocks, and I compute the average welfare, given the initial relative price distortion at the steady state. Even-
Figure 7: Average welfare relative to the case of non-distorted steady state. The probability of hitting the ZLB is around 6% in the non-distorted case.

tually, I compute the unconditional welfare gain as percentage change from the one associated with the non-distorted steady state.

It is very surprising that the unconditional welfare is decreasing in the size of the overall distortion. In other words, the unconditional expected welfare is maximal when the overall economic distortion is around zero. This happens even when the average long run inflation is approximately zero and the probability of hitting the ZLB is the greatest. Therefore, the optimal inflation rate is approximately zero.

The intuition is that although the benefit of a significantly positive inflation target relative to the non-distorted case is high conditional on a particular adverse demand
shock driving the economy to the liquidity trap with binding ZLB, as shown by the red lines in Figure 8, such a shock is rather unlikely to occur. While the economy has to incur welfare loss associated with positive inflation almost every period. In sum, the unconditional welfare declines in inflation and overall economic distortion. For example, with the overall economic distortion of 0.03 or the average long run inflation of 0.25% per year, the welfare is smaller than the non-distorted welfare about 0.01 in almost all states of the preference shock, as presented by the thick solid red line in Figure 8. Only a shock with a magnitude of at least 6.7% (or 4 standard deviations), the welfare gain relative to the non-distorted case is positive. Therefore, on average,
the unconditional welfare corresponding to the long run inflation of 0.25% per year is smaller than the one with the inflation rate of −0.02%.

Figure 9: Average welfare relative to the case of non-distorted steady state. The probability of hitting the ZLB is 9.5% in the non-distorted case.

4.5. Sensitivity analysis

For robustness check, in this section, I raise the variance of the preference shock such that the unconditional probability of hitting the ZLB increases from 6% to around 9.5% in the economy with non-distorted steady state. In this case, although the optimal inflation target is no longer zero, it is not significantly greater than zero. Figure 9 shows that the average welfare increases in inflation when inflation
is small, after a point, it decreases in inflation. The welfare is maximized when the overall economic distortion is set around 0.01, corresponding to the average long run inflation of 0.02% per year.

Figure 10: Welfare relative to the case of non-distorted steady state. The probability of hitting the ZLB is 9.5% in the non-distorted case.

Figure 10 shows that, when the size of the overall economic distortion is zero or the long run inflation is around −0.07%, a small increase in the size of distortion will make welfare relatively higher than the case of no distortion given binding ZLB, while the welfare loss is not significant conditional on no binding ZLB. On average, the unconditional welfare is larger relative to the case of non-distorted steady state.
However, when the size of overall economic distortion is large, the cost of inflation increases substantially. Although welfare gain is relatively high under a particular adverse shock driving the economy into the liquidity trap with binding ZLB, it is much lower when the economy is in normal times. On average, the unconditional welfare is smaller relative to the non-distorted case.

Billi (2011) and Coibion et al. (2012) both find strictly positive optimal long-run inflation rates because they some factors in their models that cause larger welfare losses when the ZLB binds. My model is just the canonical New Keynesian and we can view the exercise here is complementary to their work.

Specifically, Billi (2011) allows inflation indexation that results in very persistent inflation in the Phillips curve. Due to this characteristic, under a particular adverse shock that causes the ZLB to bind, his model might generate very persistent deflationary episodes associated with binding ZLB and output losses. Consequently, a binding ZLB would be more damaging in his model than in my model. Hence, the central bank is very afraid of the ZLB and, as a result, pursues much higher optimal long run inflation, about 13.4

The optimal long inflation in benchmark model of Coibion et al. (2012) is around 1.5% per year and greater than the one in this paper. The reason is that, in this paper, the central bank conducts optimal discretionary monetary policy instead of a simple Taylor rule as in the benchmark model of Coibion et al. (2012). We know from the existing literature, i.e. Nakov (2008), that when the ZLB binds, welfare losses are smaller under an optimal discretionary monetary policy than under a simple Taylor rule. So the desire for higher long-run inflation is greater in Coibion et al. (2012).

\footnote{Adding more realistic factors in the model and studying the optimal long-run inflation based on the FNL model is the author’s future research agenda.}

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More importantly, Coibion et al. (2012) allow habit formation. The habit formation makes the central bank very afraid of unstable output and inflation, especially when the ZLB binds. In addition, the risk premium shock in Coibion et al. (2012) used to cause a binding ZLB is more persistent than the preference shock in my paper. Therefore, the social planner is willing to pursue a higher inflation target and, as a result, a greater optimal long run inflation.

5. Conclusion

In this paper, I study optimal monetary policy under discretion in the presence of the occasionally binding ZLB. My contribution to the existing literature of the ZLB is twofold. First, in contrast to the existing literature assuming an efficient steady state, I allow for the case of distorted steady state due to monopoly and taxation, and study the impact of this overall distortion on optimal monetary policy in the presence of the ZLB. Second, unlike the existing literature that uses the LQ approach, I solve a FNL model. Also, I provide a comparison between the FNL and LQ models.

Using the FNL method, I find some implications for optimal discretionary monetary policy. First, under a particular adverse shock driving the economy near the ZLB, the central bank in an economy with a larger economic distortion would cut the interest rate less aggressively. Second, the social planner should choose approximately zero inflation target even when the unconditional probability of hitting the ZLB is as high as 9.5%.

I show that the FNL and LQ models produce very different optimal policy in the case of a highly distorted steady state. In this case, simplifying the FNL model using the LQ approach will result in two main inaccuracies. First, the probability of hitting the ZLB is higher in the LQ approach than in the FNL method. Second,
with a higher probability of hitting the ZLB, the nominal interest rate cut is more aggressive in the LQ model, given a particular adverse demand shock that drives the economy close to the ZLB.

There are different directions to extending my paper and to investigating the ZLB as a friction that transmits adverse shocks and amplifies macroeconomic fluctuations. First, we can use the fully nonlinear method in this paper to find out if commitment policy is still better than discretion policy when we allow for a large economic distortion. Assuming zero overall economic distortion and comparing optimal policy under discretion and commitment, as done in the literature, is misleading because the role of inflation is underestimated. Second, we can apply the FNL method to examine the interaction between the ZLB and other frictions as a transmission mechanism that magnifies macroeconomic fluctuations. Ngo (2013) studies the interaction among overborrowing, incomplete financial markets and the ZLB in generating a great recession.

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represent the views of the Federal Reserve Bank of Boston or the views of the Federal Reserve System.

7. References


Appendix A. Discretionary optimal policy under ZLB

\[
V(\Delta_{t-1}, \beta_t) = \max_{\{u_t, C_t, N_t, S_t, F_t, \pi_t, \Delta_t\}} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} - \chi \frac{N_t^{1+\eta}}{1+\eta} + \beta_t E_t V(\Delta_t, \beta_{t+1}) \right\}
\]

Subject to

\[
\frac{C_t^{-\gamma}}{(1 + i_t)} = \beta_t E_t \left[ \frac{C_{t+1}^{-\gamma}}{\Pi_{t+1}} \right] = \beta_t E_t [Z_1(\Delta_t, \beta_{t+1})]
\]

\[
F_t - C_t^{-\gamma+1} = \theta \beta_t E_t \left[ \Pi_{t+1}^{\xi-1} F_{t+1} \right] = \theta \beta_t E_t [Z_2(\Delta_t, \beta_{t+1})]
\]

\[
S_t - \frac{\chi C_t N_t^\eta}{(1 - \Phi)} = \theta \beta_t E_t \left[ \Pi_{t+1}^{\xi} S_{t+1} \right] = \theta \beta_t E_t [Z_3(\Delta_t, \beta_{t+1})]
\]

\[
\Delta_t = (1 - \theta) \left( \frac{1 - \theta \Pi_t^{\xi-1}}{1 - \theta} \right) \frac{\epsilon_t}{\epsilon_{t+1}} + \theta \Pi_t \Delta_{t-1}
\]

\[
S_t = F_t \left( \frac{1 - \theta \Pi_t^{\xi-1}}{1 - \theta} \right)^\frac{\epsilon_t}{\epsilon_{t+1}}
\]
$C_t = N_t \left( \Delta_t \right)^{-1}$

$i_t \geq 0$

I rewrite the problem in the Lagrangian form as follow:

$$L = \frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\eta}}{1+\eta} + \beta_t E_t V \left( \Delta_t, \beta_{t+1} \right)$$

$$- \lambda_{1t} \left[ \frac{C_t^{-\gamma}}{(1 + i_t)} - \beta_t E_t \left[ Z_1 \left( \Delta_t, \beta_{t+1} \right) \right] \right]$$

$$- \lambda_{2t} \left[ F_t - C_t^{-\gamma-1} - \theta \beta_t E_t \left[ Z_2 \left( \Delta_t, \beta_{t+1} \right) \right] \right]$$

$$- \lambda_{3t} \left[ S_t - \frac{\chi C_t N_t^{\eta}}{(1 - \Phi)} - \theta \beta_t E_t \left[ Z_3 \left( \Delta_t, \beta_{t+1} \right) \right] \right]$$

$$- \lambda_{4t} \left[ \Delta_t - (1 - \theta) \left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \varepsilon} \right)^{\frac{1}{1-\varepsilon}} - \theta \Pi_t^{\varepsilon} \Delta_{t-1} \right]$$

$$- \lambda_{5t} \left[ S_t - F_t \left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \varepsilon} \right)^{\frac{1}{1-\varepsilon}} \right]$$

$$- \lambda_{6t} \left[ C_t - N_t \left( \Delta_t \right)^{-1} \right]$$

The first order conditions:

$$i_t : \lambda_{1t} C_t^{-\gamma} (1 + i_t)^{-2} \leq 0 \text{ with equality if } (1 + i_t) > 1$$

$$C_t : 0 = C_t^{-\gamma} + \lambda_{1t} \gamma C_t^{-\gamma-1} / (1 + i_t) + (1 - \gamma) \lambda_{2t} C_t^{-\gamma} + \lambda_{3t} \frac{\chi N_t^{\eta}}{(1 - \Phi)} - \lambda_{6t}$$

$$N_t : 0 = -\chi N_t^{\eta} + \lambda_{3t} \frac{\chi \eta C_t N_t^{\eta-1}}{(1 - \Phi)} + \lambda_{6t} \Delta_t^{-1},$$

$$F_t : 0 = -\lambda_{2t} + \lambda_{5t} \left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \varepsilon} \right)^{\frac{1}{1-\varepsilon}}$$
\[ S_t : 0 = -\lambda_3 t - \lambda_5 t \]

\[ \pi_t : 0 = -\lambda_4 \varepsilon \left( \frac{1 - \theta \Pi^e_t^{-1}}{1 - \theta} \right)^{\frac{1}{\varepsilon}} \theta \Pi^e_t^{-2} + \lambda_4 \varepsilon \theta \Pi^e_t^{-1} \Delta_{t-1} \]

\[ + \lambda_5 \varepsilon \frac{F_t}{1 - \theta} \left( \frac{1 - \theta \Pi^e_t^{-1}}{1 - \theta} \right)^{\frac{1}{\varepsilon}} \theta \Pi^e_t^{-2} \]

\[ \Delta_t : 0 = \beta_t E_t V_\Delta (\Delta_t, \beta_{t+1}) + \lambda_{1t} \beta_t E_t [Z_{1\Delta} (\Delta_t, \beta_{t+1})] \]

\[ + \lambda_{2t} \beta_t E_t [Z_{2\Delta} (\Delta_t, \beta_{t+1})] + \lambda_{3t} \beta_t E_t [Z_{3\Delta} (\Delta_t, \beta_{t+1})] \]

\[ - \lambda_{4t} - \lambda_6 t \Delta_t^{-2} \]

and the envelope theorem:

\[ V_\Delta (\Delta_{t-1}, \beta_t) = \lambda_4 t \Pi^e_t \]

where \( Z_{i\Delta} \) denotes partial derivative of \( Z_i \) with respect to \( \Delta \). Simplifying and combining with equilibrium conditions, we obtain 13 equations with 13 variables \((R, C, N, S, F, \Pi, \Delta, \lambda_1, ..., \lambda_6)\):

1. \[ i_t : 0 = \max(\lambda_{1t} C_t^{-\gamma} (1 + i_t)^{-2}, i_t) \]

2. \[ C_t : 0 = C_t^{-\gamma} + \lambda_{1t} \gamma C_t^{-\gamma-1} / (1 + i_t) + (1 - \gamma) \lambda_{2t} C_t^{-\gamma} + \lambda_3 t \frac{\chi N_t^{\eta}}{(1 - \Phi)} - \lambda_6 t \]

3. \[ N_t : 0 = -\chi N_t^{\eta} + \lambda_3 t \frac{\chi N_t^{\eta} C_t N_t^{\eta-1}}{(1 - \Phi)} + \lambda_6 t \Delta_t^{-1}, \]

4. \[ F_t : 0 = -\lambda_{2t} + \lambda_5 t \left( \frac{1 - \theta \Pi^e_t^{-1}}{1 - \theta} \right)^{\frac{1}{\varepsilon}} \]

5. \[ S_t : 0 = -\lambda_{3t} - \lambda_5 t \]

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\begin{align*}
\pi_t : 0 &= -\lambda_{4t} \varepsilon \left(1 - \theta \Pi_t^{e-1}\right) \frac{1}{1 - \theta} + \lambda_{4t} \varepsilon \Pi_t \Delta_{t-1} \\
&\quad + \lambda_{5t} F_t \left(1 - \theta \Pi_t^{e-1}\right) \frac{1}{1 - \theta} \\
\Delta_t : 0 &= \theta \beta_t E_t \left[\lambda_{4t+1} \Pi_{t+1}^{e}\right] + \lambda_{1t} \beta_t E_t \left[Z_1 \Delta_t \left(\Delta_t, \beta_{t+1}\right)\right] \\
&\quad + \lambda_{2t} \beta_t E_t \left[Z_2 \Delta_t \left(\Delta_t, \beta_{t+1}\right)\right] + \lambda_{3t} \beta_t E_t \left[Z_3 \Delta_t \left(\Delta_t, \beta_{t+1}\right)\right] \\
&\quad - \lambda_{4t} - \lambda_{6t} N_t \Delta_t^{-2} \\
\lambda_{1t} : 0 &= \frac{C_t^{-\gamma}}{(1 + i_t^t)} - \beta_t E_t \left[Z_1 \left(\Delta_t, \beta_{t+1}\right)\right] \\
\lambda_{2t} : 0 &= F_t - C_t^{-\gamma+1} - \beta_t E_t \left[Z_2 \left(\Delta_t, \beta_{t+1}\right)\right] \\
\lambda_{3t} : 0 &= S_t - \frac{\lambda C_t N_t^{\eta}}{1 - \Phi} - \beta_t E_t \left[Z_3 \left(\Delta_t, \beta_{t+1}\right)\right] \\
\lambda_{4t} : 0 &= \Delta_t - \left(1 - \theta \right) \left(1 - \theta \Pi_t^{e-1}\right) \frac{1}{1 - \theta} - \theta \Pi_t^e \Delta_{t-1} \\
\lambda_{5t} : 0 &= S_t - F_t \left(1 - \theta \Pi_t^{e-1}\right) \frac{1}{1 - \theta} \\
\lambda_{6t} : 0 &= C_t - N_t \left(\Delta_t\right)^{-1}
\end{align*}

where

\[
\begin{align*}
Z_1 \left(\Delta_t, \beta_{t+1}\right) &= \frac{C_{t+1}^{-\gamma}}{\Pi_{t+1}^{e}} \\
Z_2 \left(\Delta_t, \beta_{t+1}\right) &= \Pi_{t+1}^{e-1} F_{t+1} \\
Z_3 \left(\Delta_t, \beta_{t+1}\right) &= \Pi_{t+1}^{e} S_{t+1} \\
Z_4 \left(\Delta_t, \beta_{t+1}\right) &= \Pi_{t+1}^{e} \lambda_{4t+1}
\end{align*}
\]
The solution of the above nonlinear system is the function of the state, \( s_t = (\Delta_{t-1}, \beta_t) \), where \( \Delta_{t-1} \) is the endogenous and \( \beta_t \) are the exogenous states with the following law of motion:

\[
\Delta_t = (1 - \theta) \left( \frac{1 - \theta \Pi_{t-1}^\varepsilon}{1 - \theta} \right)^{\frac{\varepsilon_t}{\theta \Pi_{t-1}^\varepsilon}} + \theta \Pi_{t}^\varepsilon \Delta_{t-1}
\]

\[
\ln (\beta_{t+1}) = (1 - \rho_\beta) \ln (\beta_t) + \rho_\beta \ln (\beta_t) + \varepsilon_{t+1}^\beta
\]

Appendix B. Solution method

I rewrite the above 13 functional equations with 13 unknown policy functions in a more compact form:

\[
f(s, X(s), E[Z(X(s'))], E[Z\Delta(X(s'))]) = 0
\]

Here \( f : \mathbb{R}^{3+13+4+4} \rightarrow \mathbb{R}^{13} \) is the equilibrium relationship.

where

\( s = (\Delta, \beta) \) is the current state of the economy

\( X(s) = (R(s), C(s), N(s), F(s), S(s), \Pi(s), \Delta(s), \lambda_1(s), ..., \lambda_6(s))' \) is the policy function we need to solve, \( X : \mathbb{R}^3 \rightarrow \mathbb{R}^{13} \).

\( s' \) is next period state that evolves according to the following motion equation:

\[
s' = g(s, X(s), \varepsilon) = \begin{bmatrix} \Delta' = (1 - \theta) \left( \frac{1 - \theta \Pi(s)^\varepsilon}{1 - \theta} \right)^{\frac{\varepsilon_t}{\theta \Pi(s)^\varepsilon}} + \theta \Pi(s)^\varepsilon \Delta \\ \beta' = \frac{\beta^{1-\rho_\beta}}{\beta} \beta^{\rho_\beta} \exp(\varepsilon_\beta) \end{bmatrix}
\]

\( \varepsilon = [\varepsilon_\beta] \) is the innovations of the preference shock.

\[
Z(X(s')) = \begin{bmatrix} Z_1(X(s')) = \frac{C(s')^{-\gamma}}{\Pi(s')} \\ Z_2(X(s')) = \Pi(s')^{\varepsilon - 1} F (s') \\ Z_3(X(s')) = \Pi(s')^\varepsilon S (s') \\ Z_4(X(s')) = \Pi(s')^\varepsilon \lambda_4 (s') \end{bmatrix}
\]
\[
Z_{\Delta}(X(s')) = \begin{pmatrix}
Z_{1\Delta}(X(s')) = -\frac{\gamma C(s')^{-\gamma-1}C_{\Delta}(s')}{\Pi(s')} - \frac{C(s')^{-\gamma}C_{\Delta}(s')}{\Pi(s')^2} \\
Z_{2\Delta}(X(s')) = (\varepsilon - 1) \Pi(s')^{\varepsilon-2} \Pi_{\Delta}(s') F(s') + \Pi(s')^{\varepsilon-1} F_{\Delta}(s') \\
Z_{3\Delta}(X(s')) = \varepsilon \Pi(s')^{\varepsilon-1} \Pi_{\Delta}(s') S(s') + \Pi(s')^{\varepsilon} S_{\Delta}(s') \\
Z_{4\Delta}(X(s')) = \varepsilon \Pi(s')^{\varepsilon-1} \Pi_{\Delta}(s') \lambda_4(s') + \Pi(s')^{\varepsilon} \lambda_{4\Delta}(s')
\end{pmatrix}
\]

I solve the above equilibrium relationship using a collocation method, where collocation nodes are non-equidistant in the sense that these nodes concentrate around potential kinks. First, I use equidistant nodes to solve the model using the below algorithm and find out policy function based on which I investigate potential kinds. I redistribute these nodes by clustering them around potential kinks, then resolve the model.

Below is the simplified algorithm of the collocation method:

**Step 1:** Define the space of the approximating functions and collocation nodes \( S = (S_1, ..., S_N) \), where \( N = N_{\Delta} \times N_{\beta} \) and \( (N_{\Delta} \times N_{\beta}) \) is the polynomial degrees in each dimension of the space. In this paper, I use cubic spline method where \( N_{\Delta} \times N_{\beta} \) are number of collocations nodes along each state dimension.

\[ X(s) = (\phi(s)\theta_R, \phi(s)\theta_C, \phi(s)\theta_N, \phi(s)\theta_F, \phi(s)\theta_S, \phi(s)\theta_{\Pi}, \phi(s)\theta_{\Delta}, \phi(s)\theta_{\lambda_1}, ..., \phi(s)\theta_{\lambda_6})'; \]

or \( X(s) = \phi(s)\Theta \)

where

- \( \phi(s) \) is a \( 1 \times N \) matrix of cubic spline basis functions evaluated at state \( s \in S = (S_1, ..., S_N) \).

- \( \Theta = (\theta_R; \theta_C; \theta_N; \theta_F; \theta_S; \theta_{\Pi}; \theta_{\Delta}; \theta_{\lambda_1}; ..., \theta_{\lambda_6}) \) is \( N \times 13 \) coefficient matrix that we want to approximate.

**Step 2:** Initialize the coefficient matrix \( \Theta^0 \), and set up stopping rules.

**Step 3:** At each iteration \( j \) we have a corresponding \( \Theta^j \), implement the following substep:
1. At each collocation node $s_i, s_i \in \{S_1..S_N\}$ compute $E[Z(X(s'))], E[Z_\Delta(X(s'))]$

\[
\begin{align*}
&\bullet \quad E[Z(X(s'))] = \sum_j w_j [Z(X(g(s, X(s'), e_j)))] \\
&\bullet \quad E[Z_\Delta(X(s'))] = \sum_j w_j [Z_\Delta(X(g(s, X(s'), e_j)))]
\end{align*}
\]

2. Solve for $X(s_i)$ s.t. $f(s_i, X(s_i), E[Z(X(s'))], E[Z_\Delta(X(s'))]) = 0$, I solve this complementarity problem using Newton method and semi-smooth root finding formulations laid in Miranda and Fackler (2003). I also provide an "analytical" Jacobian matrix $f_X$ (see the formula below). Both semi-smooth root finding and "analytical" Jacobian matrix help to significantly boost the speed of solving the problem and make the solver very reliable.

**Step 4:** Update $\Theta^{j+1} = \Phi^{-1}\Theta^j$, where $\Phi = (\phi(s_1), ..., \phi(s_N))'$.

**Step 5:** Check the stopping rules. If not satisfied go to Step 3; otherwise go to Step 6.

**Step 6:** Report results.

In the paper, I obtain the maximum absolute error of 1e-4 across equilibrium conditions for some states, smaller than 1e-6 for almost all other states. By using more points, I can obtain more accurate results. There is another way to solve for the policy functions. I can define the residual function, $r(s, \Theta) = f(s, X(s), E[Z(X(s'))], E[Z_\Delta(X(s'))])$ and use Newton’s method to solve $r(s, \Theta) = 0$ by updating $\Theta^{j+1} = \Theta^j - \alpha [r_{\Theta}(s, \Theta^j)]^{-1} r(s, \Theta^j)$, where $r_{\Theta}(s, \Theta^j)$ is an "analytical" Jacobian matrix. I tried this method but it is extremely slow due to the inverse of a large coefficient matrix.

---

"Analytical" because the first and second derivatives are computed using the approximating functions.
Appendix C. "Analytical" Jacobian matrix, $f_X$

\[
f_{1X} = \begin{pmatrix} -2R_t^{-3}C_t^{-\gamma}\lambda_{1t}, R_t^{-2}(\gamma)C_t^{-\gamma-1}\lambda_{1t}, 0, 0, 0, 0, 0, \lambda_2C_t^{-\gamma}R_t^{-2}, 0, 0, 0, 0, 0; \text{ if } \lambda_{1t}C_t^{-\gamma}R_t^{-2} > R_t - 1 \\
-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; \text{ if } \lambda_{1t}C_t^{-\gamma}R_t^{-2} \leq R_t - 1 \end{pmatrix}
\]

\[
f_{2X} = \begin{pmatrix} -\frac{\lambda_{1t}C_t^{-\gamma-1}}{(1+it)^2}, \frac{\gamma}{(1+it)^2}C_t^{-\gamma-1} - \lambda_{1t}(\gamma+1)C_t^{-\gamma-2} - (1 - \gamma) \gamma \lambda_{2t}C_t^{-\gamma-1}, \lambda_3, \frac{\chi N_t^{\eta-1}}{(1-\Phi)}, \ldots \\\n0, 0, 0, 0, \frac{\gamma C_t^{-\gamma-1}}{(1+it)^2}, (1 - \gamma) C_t^{\gamma}, \frac{\chi N_t^{2}}{(1-\Phi)}, 0, 0, -1; \end{pmatrix}
\]

\[
f_{3X} = \begin{pmatrix} \lambda_3, \frac{\chi \eta N_t^{\eta-1}}{(1-\Phi)}, -\chi \eta N_t^{\eta-1} + \lambda_3, \frac{\chi \eta (\eta - 1)C_t N_t^{\eta-2}}{(1-\Phi)} \lambda_3, \frac{\chi \eta N_t^{2}}{(1-\Phi)}, 0, 0, -\gamma_t, 0, 0, 0, 0, \frac{\gamma C_t N_t^{\eta-1}}{(1-\Phi)}, 0, 0, \frac{\Delta_t}{\lambda_3}; \end{pmatrix}
\]

\[
f_{4X} = \begin{pmatrix} 0, 0, 0, 0, 0, \lambda_5, \left(1 - \theta \Pi_t^{-1} \right)^{\frac{1}{\theta - 1}}, \theta \Pi_t^{-2}, \frac{1}{\theta - 1}, 0, 0, -1, 0, 0, \left(1 - \theta \Pi_t^{-1} \right)^{\frac{1}{\theta - 1}}, 0; \end{pmatrix}
\]

\[
f_{5X} = (0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, -1, 0;)
\]

\[
f_{6X} = \begin{pmatrix} 0, 0, 0, \lambda_5, \left(1 - \theta \Pi_t^{-1} \right)^{\frac{1}{\theta - 1}}, \frac{1}{\theta - 1}, 0, \lambda_4, \left(1 - \theta \Pi_t^{-1} \right)^{\frac{1}{\theta - 1}}, \frac{1}{\theta - 1}, \lambda_4 \varepsilon \Delta_t^{-1} + \ldots \\\n\lambda_5 F_t \left(1 - \theta \Pi_t^{-1} \right)^{\frac{1}{\theta - 1}}, \frac{1}{\theta - 1}, 0, 0, 0, 0, 0, -\varepsilon \left(1 - \theta \Pi_t^{-1} \right)^{\frac{1}{\theta - 1}}, \frac{1}{\theta - 1}, \varepsilon \Pi_t \Delta_t^{-1}, \ldots \\\nF_t \left(\frac{1}{\theta - 1}, \frac{1}{\theta - 1}, 0, 0, 0, 0, 0, 0, 0; \right) \end{pmatrix}
\]

\[
f_{7X} = \begin{pmatrix} \theta \beta_t \left(E_t [Z_{4\Delta t+1}] + \lambda_{1t} E_t [Z_{1\Delta t+1}] / \theta + \lambda_2 E_t [Z_{2\Delta t+1}] + \lambda_3 E_t [Z_{3\Delta t+1}] \right) + \frac{2\lambda_t N_t}{\Delta_t}, \theta \beta_t E_t [Z_{1\Delta t+1}], \theta \beta_t E_t [Z_{2\Delta t+1}], \theta \beta_t E_t [Z_{3\Delta t+1}], -1, 0, -\frac{N_t}{\Delta_t}; \right) \end{pmatrix}
\]

\[
f_{8X} = \begin{pmatrix} -\frac{C_t^{-\gamma}}{(1+it)^2}, \frac{-\gamma C_t^{-\gamma-1}}{(1+it)^2}, 0, 0, 0, 0, \beta_t E_t [Z_{1\Delta t+1}], 0, 0, 0, 0, 0; \right) \end{pmatrix}
\]

\[
f_{9X} = \begin{pmatrix} 0, -\left(-\gamma + 1\right) C_t^{-\gamma}, 0, 1, 0, 0, 0, \theta \beta_t E_t [Z_{2\Delta t+1}], 0, 0, 0, 0, 0; \right) \end{pmatrix}
\]

\[
f_{10X} = \begin{pmatrix} 0, -\frac{\chi N_t^{\eta}}{(1-\Phi)}, -\frac{\chi N_t^{\eta}}{(1-\Phi)} , 0, 1, 0, 0, \theta \beta_t E_t [Z_{3\Delta t+1}], 0, 0, 0, 0, 0; \right) \end{pmatrix}
\]
\[ f_{11x} = \left(0, 0, 0, 0, 0, \left(1 - \theta \Pi^{-1}_t \right)^{\frac{1}{\varepsilon}} \varepsilon \theta \Pi^{-2}_t - \theta \varepsilon \Pi^{-1}_t \Delta_{t-1}, 1, 0, 0, 0, 0, 0; \right) \]

\[ f_{12x} = \left(0, 0, 0, -\left(1 - \theta \Pi^{-1}_t \right)^{\frac{1}{\varepsilon}}, 1, -F_t \left(1 - \theta \Pi^{-1}_t \right)^{\frac{1}{\varepsilon}} \theta \Pi^{-2}_t \right) \]

\[ f_{13x} = \left(0, 1, -\frac{1}{\Delta_t}, 0, 0, 0, \frac{N_t}{\Delta^2_t}, 0, 0, 0, 0, 0; \right) \]

where

\[ Z_{1\Delta t+1} = -\frac{\gamma C_{t+1}^{-1} C_{\Delta t+1}}{\Pi_{t+1}} - \frac{C_{t+1}^{-1} \Pi_{\Delta t+1}}{\Pi^2_{t+1}} \]

\[ Z_{2\Delta t+1} = (\varepsilon - 1) \Pi_{t+1}^{\varepsilon-2} \Pi_{\Delta t+1} F_{t+1} + \Pi_{t+1}^{\varepsilon-1} F_{\Delta t+1} \]

\[ Z_{3\Delta t+1} = \varepsilon \Pi_{t+1}^{\varepsilon-1} \Pi_{\Delta t+1} S_{t+1} + \Pi_{t+1}^{\varepsilon} S_{\Delta t+1} \]

\[ Z_{4\Delta t+1} = \varepsilon \Pi_{t+1}^{\varepsilon-1} \Pi_{\Delta t+1} \lambda_{t+1} + \Pi_{t+1}^{\varepsilon} \lambda_{\Delta t+1} \]

\[ Z_{1\Delta \Delta t+1} = \frac{\gamma (\gamma + 1) C_{t+1}^{\gamma-1} C^2_{\Delta t+1}}{\Pi_{t+1}} - \gamma C_{t+1}^{\gamma-1} \left(\frac{C_{\Delta t+1}}{\Pi_{t+1}} - \frac{C_{\Delta t+1} \Pi_{\Delta t+1}}{\Pi^2_{t+1}}\right) \]

\[ \frac{\gamma C_{t+1}^{\gamma-1} C_{\Delta t+1} \Pi_{\Delta t+1}}{\Pi^2_{t+1}} - C_{t+1}^{\gamma} \left(\frac{\Pi_{\Delta t+1}}{\Pi^2_{t+1}} - \frac{2 \Pi^3_{\Delta t+1}}{\Pi^2_{t+1}}\right) \]

\[ Z_{2\Delta \Delta t+1} = (\varepsilon - 1) (\varepsilon - 2) \Pi_{t+1}^{\varepsilon-3} \Pi^3_{\Delta t+1} F_{t+1} + (\varepsilon - 1) \Pi_{t+1}^{\varepsilon-2} (\Pi_{\Delta t+1} F_{t+1} + \Pi_{t+1} F_{\Delta t+1}) \]

\[ + (\varepsilon - 1) \Pi_{t+1}^{\varepsilon-2} \Pi_{\Delta t+1} F_{t+1} + \Pi_{t+1}^{\varepsilon-1} F_{\Delta t+1} \]

\[ Z_{3\Delta \Delta t+1} = \varepsilon (\varepsilon - 1) \Pi_{t+1}^{\varepsilon-2} \Pi^2_{\Delta t+1} S_{t+1} + \varepsilon \Pi_{t+1}^{\varepsilon-1} (\Pi_{\Delta t+1} S_{t+1} + \Pi_{t+1} S_{\Delta t+1}) \]

\[ + \varepsilon \Pi_{t+1}^{\varepsilon-1} \Pi_{\Delta t+1} S_{\Delta t+1} + \Pi_{t+1}^{\varepsilon} S_{\Delta \Delta t+1} \]
Appendix D. Simplified LQ model of the FNL model

The LQ version of the FNL model is:

\[
\begin{align*}
Z_{3\Delta t+1} &= \varepsilon (\varepsilon - 1) \Pi_{t+1}^{\varepsilon-2} \Pi_{t+1}^{\Delta} \lambda_{t+1} + \varepsilon \Pi_{t+1}^{\varepsilon-1} \Pi_{t+1}^{\Delta} \lambda_{t+1} + \Pi_{t+1}^{\varepsilon} \lambda_{t+1} \\
& \quad + \varepsilon \Pi_{t+1}^{\varepsilon-1} \Pi_{t+1}^{\Delta} \lambda_{t+1} + \Pi_{t+1}^{\varepsilon} \lambda_{t+1}
\end{align*}
\]

\[
\text{Max} \left\{ -\Omega E_0 \sum_{t=0}^{\infty} \beta^t \{ \hat{\pi}^2_t + \lambda (x_t - x^*)^2 \} \right\}
\]

subject to

\[
x_t = E_t x_{t+1} - \frac{1}{\gamma} \left[ \hat{i}_t - E_t \hat{\pi}_{t+1} \right] + \frac{1}{\gamma} \left[ -\hat{\beta}_t \right]
\]

\[
\hat{\pi}_t = \beta E_t [\hat{\pi}_{t+1}] + \kappa x_t
\]

\[
\hat{\beta}_t = \ln (\beta_t) - \ln \beta = \rho \hat{\beta}_{t-1} + \varepsilon_{\beta,t}
\]

\[
\hat{i}_t \geq \log (\beta)
\]

where

\[
\hat{\pi}_t = \log (1 + \pi_t)
\]

\[
\hat{i}_t = \log (1 + i_t) - \log (1/\beta)
\]

\[
x_t = (\log (Y_t) - \log (Y)) - (\log (Y^f_t) - \log (Y^f))
\]

\[
\Omega = -\frac{1}{\gamma + \eta} \frac{\varepsilon}{\gamma + \eta} \frac{\theta}{(1 - \theta)(1 - \theta \beta)} \theta^{\gamma + 1}
\]

\[
\lambda = \frac{(\gamma + \eta)(1 - \theta)(1 - \theta \beta)}{\varepsilon} \frac{\kappa}{\theta} = \frac{\kappa}{\varepsilon}
\]

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\[ x^* = \frac{\Phi}{\gamma + \eta} \]
\[ \Phi = 1 - (1 - \tau_w) \left( 1 - \varepsilon^{-1} \right) \]

It is not difficult to show that the steady state values of this model are as follows:

\[ \overline{\pi} = \frac{\lambda \kappa}{\kappa^2 + \lambda (1 - \beta) x^*} \]

\[ \overline{\bar{i}} = \overline{\pi} \]

\[ \bar{x} = \frac{(1 - \beta) \overline{\pi}}{\kappa} \]