A Tale of Two Price Adjustments: Does Calvo Meet Rotemberg at the ZLB?

Jianjun Miao  
*Department of Economics, Boston University*

Phuong V. Ngo*  
*Department of Economics, Cleveland State University*

July 2014

Abstract

This paper examines the difference between the most popular models of price adjustments, the Calvo model and the Rotemberg model, in a fully nonlinear dynamic stochastic general equilibrium (DSGE) framework with an occasionally binding zero lower bound (ZLB) on the nominal interest rate. Although the two models are equivalent at the first-order approximation under a specific set of parameters, they are very different in the fully nonlinear framework. Specifically, we find that (i) under an adverse shock driving the economy close to the liquidity trap, the interest rate cut is greater in the Calvo model than in the Rotemberg model, and, as a result, the ZLB is more likely to bind in the Calvo model than in the Rotemberg model; (ii) the government spending multiplier is smaller than one on impact in the Rotemberg model even when the ZLB binds; (iii) while the government spending multiplier is also smaller than one on impact in the Calvo model when the ZLB does not bind, it can be greater than one when the ZLB binds; (iv) the cumulative government spending multiplier can be larger than one in the Calvo model, but it is always less than one in the Rotemberg model; and (v) the relative price dispersion plays a key role in explaining the difference between the two models.

**JEL classification:** C61, E31, E52, E62.  
**Keywords:** Calvo price adjustments, Rotemberg price adjustments, Taylor rule, ZLB, global methods, full nonlinearity.

*Corresponding author. Tel. +1 617 347 2706. Email: p.ngo@csuohio.edu (Phuong Ngo).*
1 Introduction

In the New Keynesian literature, the two most popular stylized pricing behaviors are the Calvo type and the Rotemberg type. According to the Calvo type, in each period firms face an exogenous fixed probability of adjusting their prices, while according to the Rotemberg type, firms can adjust their prices whenever they want after paying adjustment costs. It is well-known that a model with Calvo price adjustments, or the Calvo model, and a model with Rotemberg price adjustments, or the Rotemberg model, are equivalent at the first-order approximation under a specific set of parameters. However, whether or not these two models produce the same results in a fully nonlinear dynamic stochastic general equilibrium (DSGE) framework with occasionally binding zero lower bound (ZLB) on the nominal interest rate is not well understood or documented.

This paper fills the gap by using a fully nonlinear method to solve the two models and answer the following questions: (i) do the models generate the same policy functions and imply the same interest rate dynamics before, during, and after the ZLB? (ii) do they generate the same distribution of inflation and interest rates in the long run? (iii) are government spending multipliers the same in the two models during normal times when the ZLB is not binding and during deep recessions when the ZLB binds? If not, how different are they?

Solving the two models that are equivalent at the first-order approximation using a fully nonlinear method we find that: (i) under an adverse shock driving the economy close to the liquidity trap, the interest rate cut is greater in the Calvo model than in the Rotemberg model, and, as a result, the ZLB is more likely to bind in the Calvo model than in the Rotemberg model; (ii) the government spending multiplier is smaller than one on impact in the Rotemberg model even when the ZLB binds; (iii) while the government spending multiplier is also smaller than one on impact in the Calvo model when the ZLB does not bind, it can be greater than one when the ZLB binds; and (iv) the cumulative government
spending multiplier can be larger than one in the Calvo model, but it is always less than one in the Rotemberg model.

The main reason for the discrepancy lies in the state variable called the relative price dispersion and its role as a negative technology shock. It is well-known that the higher the relative price dispersion, the less efficient the economy. While the relative price dispersion is always equal to one in the Rotemberg model, it can be substantially greater than one in the Calvo model, especially when the ZLB binds.

The intuition is that when the ZLB binds, the central bank is not able to obtain inflation that close to the target. In fact, there is a sharp decline in output and inflation, resulting in a substantial increase in the relative price dispersion. This increase in the relative price dispersion, in turn, causes output and inflation to fall more, and so on. Eventually, the recession is much more severe in the Calvo model than in the Rotemberg model, resulting in a more effective government spending in the Calvo model than in the Rotemberg model.

In the existing literature on the government spending multiplier under the ZLB, many researchers use the linearized version of the fully nonlinear models presented in this paper, including Eggertsson and Krugman [2012] and Woodford [2011]. The difference between the Calvo and Rotemberg models in the fully nonlinear framework, as found in this paper, requires us to be more cautious in term of using and interpreting the linearized model for policy analysis.

In addition, there is a common belief among economists and policymakers that New Keynesian models with sticky prices always generate a larger-than-one government spending multiplier when the ZLB binds. The intuition is that when the ZLB is binding, government spending expansion does not crowd out private consumption and investment as the interest rate remains constant at the ZLB; equally importantly, government spending can generate higher inflation that, in turn, lowers the real interest rate, promoting consumption and output. It is worth noting that this belief is drawn from models that incorporate the Calvo
price adjustments, including Christiano et al. [2011], Woodford [2011], Fernandez-Villaverde et al. [2012], and many others.

Therefore, this paper challenges the conventional belief by arguing that there are some classes of New Keynesian models with sticky prices that actually produce a less-than-one government spending multiplier even when the ZLB binds; specifically, the class of models with Rotemberg price adjustments. As a result, the conclusion that the expansion of government spending is always effective, or that the government spending multiplier is always greater than one when the ZLB binds in a sticky price model, needs further investigation.

The remainder of this paper is organized as follows. Section 2 presents the structure of the Calvo and Rotemberg models and explains why they are the same at the first-order approximation given a specific set of parameters. Section 3 shows the benchmark calibration and solution method, and Section 4 shows results from the two models regarding interest rates, inflation, and government spending multipliers. Section 5 contains some sensitivity analyses of the main findings with respect to some important assumptions. Section 6 concludes.

2 Models

The economic structure in this paper presents two key New Keynesian features. Specifically, intermediate goods producers are monopolistic competitors. In addition, the producers reset their prices infrequently. The specific structures of the models are as follows.

2.1 Households

The representative household maximizes his total expected discounted flow utilities:

\[
Max E_t \left\{ \left( \frac{C_{t+1}^{1-\gamma}}{1-\gamma} + \chi \frac{N_{t+1}^{1+\eta}}{1+\eta} \right) + \sum_{j=1}^{\infty} \left\{ \beta^j \left( \prod_{k=0}^{j-1} \beta_{t+k} \right) \left( \frac{C_{t+j}^{1-\gamma}}{1-\gamma} + \chi \frac{N_{t+j}^{1+\eta}}{1+\eta} \right) \right\} \right\}
\]  

(1)
subject to the budget constraint:

\[ C_t + B_t = w_t N_t + B_{t-1} \left( \frac{1 + i_{t-1}}{1 + \pi_t} \right) + \int_0^1 D_t(i) di + T_t \]  \hspace{1cm} (2)

where \( C, N \) are composite consumption and total labor; \( B, D, T \) denote real bonds, dividends, and lump sum transfers; \( i, \pi \) are the nominal interest rate and the inflation rate, respectively; \( w \) is the real wage; \( \gamma, \eta, \chi \) are the risk aversion parameter, the inverse wage elasticity of labor with respect to wages, and the steady state labor determining parameter; \( \beta_t \) is the shock to the subjective time discount factor \( \beta \), or the preference shock, that follows an AR(1) process:

\[ \ln \left( \beta_{t+1} \right) = \rho_{\beta} \ln (\beta_t) + \varepsilon_{\beta,t+1}, \text{ where } \beta_t \text{ is given.} \]  \hspace{1cm} (3)

where \( \rho_{\beta} \in (0, 1) \) is the persistence of the preference shock; and \( \varepsilon_{\beta,t} \) is the innovation of the preference shock with mean 0 and variance \( \sigma_{\beta}^2 \). The preference shock is a reduced form of more realistic forces that can drive the nominal interest rate to the ZLB.\(^1\)

The optimal choices of the household give rise to the labor supply:

\[ \chi N_t^\rho C_t^\gamma = w_t \]  \hspace{1cm} (4)

and the Euler equation:

\[ E_t \left( M_{t,t+1} \left( \frac{1 + i_t}{1 + \pi_{t+1}} \right) \right) = 1 \]  \hspace{1cm} (5)

---

\(^1\)Guerrieri and Lorenzoni [2011] model debt limit and household heterogeneity in labor productivity. They show that an exogenous decline in the debt limit acts as an increase in the subjective discount factor in our representative agent model. The decline in the debt limit causes future consumption to be more volatile because with a lower debt limit households will be less able to insure their consumption against risks. Therefore, the savers will save more and the borrowers will borrow less due to precautionary savings.
where the stochastic discount factor is:

\[ M_{t,t+1} = \beta_t \beta_t \left( \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \right) \] (6)

2.2 Final goods producers

To produce the composite final goods, the final goods producers buy and aggregate a variety of intermediate goods using a CES technology. Their cost-minimization problem is given below.

\[ \min \int_0^1 P_t(i) Y_t(i) \, di \quad \text{s.t.} \quad Y_t = \left( \int_0^1 Y_t(i)^{1-\varepsilon} \, di \right)^{\frac{1}{1-\varepsilon}} \] (7)

where \( P_t(i) \) and \( Y_t(i) \) are the price and the amount of intermediate goods \( i \in [0, 1] \); and \( \varepsilon \) is the elasticity of substitution among intermediate goods.

The optimal condition gives rise to the demand for the intermediate goods \( i, Y_{i,t} \), and the aggregate price level, \( P_t \), below:

\[ Y_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} Y_t \] (8)

\[ P_t = \left( \int P_t(i)^{1-\varepsilon} \, di \right)^{\frac{1}{1-\varepsilon}} \] (9)

2.3 Intermediate goods producers

There is a mass one of intermediate goods producers that are monopolistic competitors. Given its price \( P_{i,t} \) and demand \( Y_{i,t} \), firm \( i \in [0, 1] \) chooses labor that

\[ \min \{ w_t N_{i,t} \} \quad \text{s.t.} \quad Y_{i,t} = A_t N_{i,t} \] (10)
where $A$ denotes the technology shock; $\rho_A \in (0, 1)$ is the persistence of the technology shock; and $\varepsilon_{At}$ is the innovation of the technology shock with mean 0 and variance $\sigma^2_A$.

Let $\varphi_{i,t}$ be the Lagrange multiplier with respect to the production. The first-order condition gives the same marginal cost, $\varphi_t$, to all firms:

$$\varphi_t = \varphi_{i,t} = \frac{w_t}{A_t} \quad (11)$$

### 2.4 Price adjustments

In the New Keynesian literature, the two most popular stylized price adjustments are the Calvo type and the Rotemberg type. As previously stated, according to the Calvo type, in each period firms face an exogenous fixed probability of adjusting their prices, while according to the Rotemberg type, firms can adjust their prices whenever they want after paying adjustment costs in terms of final goods.

#### 2.4.1 The Calvo type

In each period, an intermediate goods firm $i$ keeps and charges its previous price with probability $\theta$ and resets its price with probability $(1 - \theta)$. Whenever firm $i$, for $i \in [0, 1]$, has a chance to reset its price, it chooses the new price to solve:

$$Max_{P_{i,t}} \mathbb{E}_{t} \left\{ \sum_{j=0}^{\infty} \left\{ \theta^j M_{t,t+j} \left[ \frac{P_{i,t}}{P_{t+j}} - \varphi_{t+j} \right] Y_{i,t+j} \right\} \right\} \quad (12)$$

subject to its demand in equation (8) and

$$M_{t,t+j} = 1 \text{ if } j = 0$$
$$M_{t,t+j} = \prod_{s=0}^{j-1} M_{t+s,t+s+1} \text{ for } j \geq 1.$$
The optimal relative price, \( p_t^* = \frac{P_{i,t}^*}{P_t} \), is the same for all firms who have a chance to reset their prices today:

\[
p_t^* = \frac{P_{i,t}^*}{P_t} = \frac{\left(\frac{\varepsilon}{\varepsilon - 1}\right) E_t \left\{ \sum_{j=0}^{\infty} \left\{ \theta^j M_{t,t+j} \left( \frac{P_{t+j}}{P_t} \right) \varepsilon \right\} \right\}}{E_t \left\{ \sum_{j=0}^{\infty} \left\{ \theta^j M_{t,t+j} \left( \frac{P_{t+j}}{P_t} \right) \varepsilon - 1 \right\} \right\}}
\]

(13)

With some manipulation, we can rewrite the optimal pricing rule as below:

\[
p_t^* = \frac{S_t}{F_t}
\]

(14)

where \( S_t, F_t \) are written in the following recursive forms:

\[
S_t = \left(\frac{\varepsilon}{\varepsilon - 1}\right) C_t^{-\gamma} Y_t \varphi t + \theta \beta E_t \left[ \beta_t \Pi_t^{\gamma-1} S_{t+1} \right]
\]

(15)

\[
F_t = C_t^{-\gamma} Y_t + \theta \beta E_t \left[ \beta_t \Pi_t^{\gamma-1} F_{t+1} \right]
\]

(16)

and \( \Pi = (1 + \pi) \) is gross inflation.

Combining (15) with (4) and (11), we obtain:

\[
S_t = \frac{\lambda Y_t N_t^\eta}{(1 - \Phi)} + \theta \beta E_t \left[ \beta_t \Pi_t^{\gamma-1} S_{t+1} \right]
\]

(17)

where

\[
\Phi = 1 - (1 - \varepsilon^{-1})
\]

(18)

### 2.4.2 The Rotemberg type

The intermediate goods firms adjust their prices according to a quadratic adjustment cost. In other words, firms are allowed to adjust their prices whenever they want as long as they
pay an adjustment cost in terms of final goods. The problem of firm \( i \), for \( i \in [0, 1] \), is given as follows:

\[
\max_{\{P_{i,t}\}} \mathbb{E}_t \sum_{j=0}^{\infty} \left \{ M_{t,t+j} \left [ \left ( \frac{P_{i,t+j}}{P_{t+j}} - \varphi_t \right ) Y_{i,t+j} - \frac{\varphi}{2} \left ( \frac{P_{i,t+j}}{P_{i,t+j-1}} - 1 \right )^2 Y_{t+j} \right ] \right \}
\]

subject to its demand in equation (8) and the same initial price for all firms:

\[
P_{i,0} = P_0
\]

where \( \varphi \) is the adjustment cost parameter.

The optimal pricing rule then gives rise to the following condition:

\[
\left ( 1 - \varepsilon + \frac{\varepsilon \frac{w_t}{A_t} - \varphi \pi_t (1 + \pi_t)}{M_{t,t+1}} \right ) Y_t + \varphi \mathbb{E}_t [M_{t,t+1} \pi_{t+1} (1 + \pi_{t+1}) Y_{t+1}] = 0
\]

where \( M_{t,t+1} \) is the stochastic discount factor defined in equation (6).

### 2.5 Macroeconomic policy

**Monetary policy:** The central bank conducts monetary policy using a simple Taylor rule as follows:

\[
\left ( \frac{1 + i_t}{1 + i} \right ) = \left ( \frac{Y_t}{Y} \right )^\phi_y \left ( \frac{1 + \pi_t}{1 + \pi} \right )^\phi_{\pi}
\]

\[
i_t \geq 0
\]

where \( \pi, i, Y \) are the targeted inflation, the steady state nominal interest rate, and the steady state output, respectively.

Equation (23) implies that the nominal interest rate is not allowed to be negative. This
is the key condition in the ZLB literature.

**Fiscal policy:** the government spending:

\[ G_t = Y_t \left( S_G g_t \right) \]

where \( S_G \) denotes the steady state share of government spending in total output; \( g_t \) denotes the government spending shock that follows an AR(1) process:

\[ \ln g_t = \rho_g \ln g_{t-1} + \varepsilon_{gt} \]

where \( \rho_g \in (0, 1) \) is the persistence of the government spending shock; and \( \varepsilon_{gt} \) is the innovation of the government spending shock with mean 0 and variance \( \sigma_g^2 \).

### 2.6 Aggregate condition

#### 2.6.1 The Calvo model

In the Calvo model, aggregate output satisfies:

\[ C_t + G_t = Y_t \quad (24) \]

\[ Y_t = \frac{A_t N_t}{\Delta_t} \quad (25) \]

where \( \Delta_t \) is called the relative price dispersion and is defined as:

\[ \Delta_t = \int \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} di \quad (26) \]

or, in a recursive form:

\[ \Delta_t = \theta \Pi_t^\varepsilon \Delta_{t-1} + (1 - \theta) \left( p_t^* \right)^{-\varepsilon} \quad (27) \]
We write the price level (9) in a recursive form and divide both sides by $P_t$ to obtain the optimal relative price:

$$p_t^* = \left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \theta} \right)^{\frac{1}{1-\varepsilon}}$$  \hspace{1cm} (28)

Plugging this optimal relative price into the relative price dispersion equation (27), we have:

$$\Delta_t = (1 - \theta) \left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \theta} \right)^{\frac{\varepsilon}{1-\varepsilon}} + \theta \Pi_t \Delta_{t-1}$$  \hspace{1cm} (29)

### 2.6.2 The Rotemberg model

In the Rotemberg model, the aggregate condition is given below:

$$C_t + G_t + \frac{\varphi}{2} \pi_t^2 Y_t = Y_t$$  \hspace{1cm} (30)

$$Y_t = A_t N_t$$  \hspace{1cm} (31)

### 2.7 Equilibrium and the case of equivalence

The equilibrium for each model consists of the path of prices and allocation that solves the system of nonlinear equations summarized in Appendix A.

As shown in Appendix B, at the first-order approximation around the steady state with zero inflation, the Calvo model becomes:

$$i_t = \max \left\{ 0, \phi_y x_t + \phi_\pi \pi_t + \log (1/\beta) + \phi_y \log (Y_t^f) \right\}$$

$$x_t = E_t x_{t+1} - \frac{1}{\gamma} (i_t - E_t \pi_{t+1}) + \frac{1}{\gamma} n_t$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t$$
and the Rotemberg model becomes:

\[
\begin{align*}
  i_t &= \max\left\{0, \phi_y x_t + \phi_x \pi_t + \log(1/\beta) + \phi_y \log\left(Y_t^f\right)\right\} \\
  x_t &= E_t x_{t+1} - \frac{1}{\gamma}\left(i_t - E_t \pi_{t+1}\right) + \frac{1}{\gamma} r_t^n \\
  \pi_t &= \beta E_t \pi_{t+1} + \tilde{\kappa} x_t 
\end{align*}
\]

where \(Y_t^f = \left(\frac{1}{\mu \chi (1-S_g \exp(gt))^{\gamma}}\right)^{\frac{1}{\eta+\gamma}}\) is the flexible price output; \(x_t = \log(Y_t) - \log\left(Y_t^f\right)\) is output gap; and

\[
\begin{align*}
  r_t^n &= -\beta_t + \frac{\gamma (1 + \eta)}{\eta + \gamma} (\rho_A - 1) A_t - \frac{\gamma \eta}{\eta + \gamma} \left(\frac{S_g}{1 - S_g}\right) (\rho_G - 1) g_t \\
  \text{or } r_t^n &= -\beta_t + \frac{\gamma (1 + \eta)(1 - S_g)}{\eta(1 - S_g) + \gamma} (\rho_A - 1) A_t - \frac{\gamma \eta S_g (1 - S_g)}{\eta(1 - S_g) + \gamma} (\rho_G - 1) g_t \\
  \kappa &= \frac{(\gamma + \eta) (1 - \theta) (1 - \theta \beta)}{\theta} \\
  \tilde{\kappa} &= \frac{(\gamma + \eta) (\varepsilon - 1)}{\varphi} 
\end{align*}
\]

It is obvious that by choosing the price adjustment cost parameter \(\varphi = \frac{(\varepsilon - 1) \theta}{(1 - \theta)(1 - \theta \beta)}\), then \(\kappa\) equals \(\tilde{\kappa}\) and the two models are the same at the first-order approximation. Therefore, results from the two models are the same even in the presence of the ZLB.

3 Solution method and calibration

3.1 Solution method

The solution method is the same as the one used in Ngo [2014]. We solve both the Calvo model and the Rotemberg model using a collocation method associated with cubic spline
basis functions to capture kinks due to the ZLB. At each collocation node, we solve a comple-
mentarity problem using the Newton method and the semi-smooth root-finding algorithm as
described in Miranda and Fackler [2002]. We also provide an "analytical" Jacobian matrix
computed from the approximating functions. Moreover, we write our code using a paral-
lel computing method that allows us to split up a large number of collocation nodes into
smaller groups that then are assigned to different processors to be solved simultaneously.
These computational characteristics help to significantly reduce computational costs.

3.2 Calibration

We calibrate the parameters on the basis of the observed data and other studies. The
quarterly subjective discount factor, $\beta$, is 0.993, corresponding to the real interest rate
of 2.8% per annum. The constant relative risk aversion, $\gamma$, is 1, corresponding to a log
utility function with respect to consumption. This utility function is commonly used in the
literature of business cycles. The labor supply elasticity with respect to wages is set at 2, or
$\eta = 1/2$, which is in the range of the numbers reported; $\chi = 1$, is normalization of the hours
worked that is almost irrelevant to the results of the paper. The elasticity of substitution
among differentiated intermediate goods, $\epsilon$, is 10, corresponding to a 11% net markup that
is in the range found by Diewert and Fox [2008].

The probability of keeping prices unchanged, $\theta$, in the Calvo model is calibrated to be
0.75, resulting in the average duration of four quarters with prices being kept unchanged.
The price stickiness parameter is in line with the empirical evidence reported by Nakamura
and Steinsson [2008]. The price adjustment cost parameter in the Rotemberg model, $\varphi$, is
calibrated to be 106 to ensure that the two models are equivalent at the first-order approxi-
mation around the steady state with zero inflation. This value also implies that the average

---

2 See Appendix for the "analytical" Jacobian matrix.
3 See Appendix for how to solve the model in detail, including the error reported from checking the solution.
duration of keeping prices unchanged is four quarters, as in the Calvo model.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Quarterly discount factor</td>
<td>0.993</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>CRRA parameter</td>
<td>1</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Inverse labor supply elasticity</td>
<td>1/2</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Monopoly power</td>
<td>10</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Probability of keeping prices unchanged in the Calvo model</td>
<td>0.75</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>Price adjustment cost parameter in the Rotemberg model</td>
<td>106</td>
</tr>
<tr>
<td>$\pi$</td>
<td>Inflation target</td>
<td>0</td>
</tr>
<tr>
<td>$\phi_\pi$</td>
<td>Weight of inflation target in the Taylor rule</td>
<td>2.5</td>
</tr>
<tr>
<td>$\phi_y$</td>
<td>Weight of output target in the Taylor rule</td>
<td>0</td>
</tr>
<tr>
<td>$S_G$</td>
<td>Share of the government spending at the steady state</td>
<td>0.2</td>
</tr>
<tr>
<td>$\sigma_\beta$</td>
<td>Standard deviation of preference shocks (percent)</td>
<td>0.3</td>
</tr>
<tr>
<td>$\rho_\beta$</td>
<td>AR-coefficient of preference shocks</td>
<td>0.8</td>
</tr>
<tr>
<td>$\sigma_G$</td>
<td>Standard deviation of government spending shocks (percent)</td>
<td>0.25</td>
</tr>
<tr>
<td>$\rho_G$</td>
<td>AR-coefficient of government spending shocks</td>
<td>0.8</td>
</tr>
</tbody>
</table>

The inflation target is set at zero. We choose this value to make sure that, together with $\theta = 0.75$ and $\varphi = 106$, the Calvo model and the Rotemberg model are equivalent at the first-order approximation. The main purpose of the paper is to compare the two models using a fully nonlinear method. Hence, it is crucial to make sure that they are the same at the first-order approximation. The parameters in the Taylor rule, $\phi_\pi = 2.5$ and $\phi_y = 0$, are in the range of the empirical studies.\textsuperscript{4} The share of the government spending in GDP, $S_G$, is in the range of the empirical studies.\textsuperscript{4} We also did some exercises using different values: $\phi_\pi = 1.5$ and $\phi_y = 0.25$. In this exercise, we modified the Taylor rule in the Rotemberg model such that the central bank stabilizes net output, defined as consumption plus government spending, instead of total output that includes adjustment costs. By doing so, we make sure that the Taylor rule is the same in both models. However the results are not very different.
is 20%.

The persistence of the preference shock and the government spending shock, $\rho_\beta$ and $\rho_g$, are calibrated to be 0.8, which is consistent with the persistence of the natural interest rate, as in Adam and Billi [2007]. In our analysis related to government spending multipliers, we actually solve the model with different values for the persistence of the government spending shock. As the government spending shock is not the main factor in driving the nominal interest rate to the ZLB, we choose its standard deviation of 0.25% as in Fernandez-Villaverde et al. [2012]. The remaining difficulty is to determine how large the standard deviation of the preference shock actually is.

In this paper, we calibrate the standard deviation of the preference shock to be 0.3% per quarter, which enables the model to generate the unconditional probability of hitting the ZLB at around 5.6%. This value is a little small compared to the fact that the nominal interest in the U.S. has been at the ZLB since December 2008 and that it is projected to be at the bound until mid-2015. However, this value is in the middle of the range 5%–6% found in the empirical studies before the last financial crisis, as discussed in Fernandez-Villaverde et al. [2012].

4 Results

4.1 Policy function

Figure 1 shows the policy function at each positive state of the preference that is presented as an annualized percentage deviation from the steady state, given no government spending shock and the initial relative price dispersion at the steady state in the Calvo model.\(^5\) The solid blue lines represent the results from the Calvo model, while the dashed green lines

\(^5\)We do not show results for negative preference shocks because these shocks cause the real interest rate to increase and the central bank has no difficulty raising the nominal interest rate to address these shocks.
represent those from the Calvo model without the ZLB imposed. The dot-dashed red lines show the results from the Rotemberg model.

When there is a positive preference shock, households value their future consumption more. In other words, they are more patient so they tend to save more and consume less today, putting downward pressure on output and the price level. To restore consumption and output, we need a lower real interest rate. If the central bank was not restrained by the ZLB, it could adjust the nominal interest rate so that the actual real interest rate is the same as the natural real rate.

However, because the ZLB is imposed, a big positive preference shock causes the ZLB to bind. As a result, the actual real rate will be larger than the natural real rate because the nominal interest rate cannot be negative. In general, the results from the two models have the same characteristics as those in Adam and Billi [2007] and Ngo [2014].

First, in the absence of the ZLB, the central bank can almost achieve the target inflation and output by adjusting the nominal interest rate as much as possible, even to a negative number. Note that the monetary policy is a simple Taylor rule in both the Calvo model and the Rotemberg model. Thus, the government cannot obtain the target output and inflation perfectly as it could using optimal monetary policy, as in Ngo [2014].

Second, when the ZLB is present, the central bank cannot stabilize output and inflation under shocks that cause the ZLB to bind. As seen in Panels A and C of Figure 1, when the nominal interest rate hits the ZLB, the real interest rate is higher than the natural real rate in both models, which is close to the real interest rate in the case without the ZLB. As a result, consumption, output, and hours worked fall substantially, as in Panels D, E, and F of Figure 1.

Third, the central bank cuts the nominal interest rate more aggressively, especially when the economy is near the ZLB, in the model with the stochastic ZLB than in the model without the ZLB. The aggressiveness occurs due to the risk of falling into the liquidity trap
Figure 1: Policy functions, the initial relative price dispersion and the government shock are at the steady state, $\Delta t_{-1} = 1, g = 0$. The preference shock is presented as an annualized percentage deviation from the steady state.
associated with deflation that forces the central bank to cut the interest rate more than it would without the risk.\textsuperscript{6}

Note that in both models we see substantial output losses and deflation associated with a binding ZLB. These results seem counterfactual because we did not observe such deflation in the U.S. during the Great Recession. This disparity occurs due to the fact that the target inflation here is zero. If we solve the model with a positive inflation target, it is likely that the economy can be pushed into a deep recession associated with moderate inflation that is smaller than the target inflation. Ngo [2014] explains that when the target inflation is high, the expected inflation is high as well. A sharp decline in output may cause a substantial downward pressure on the price level. However, the downward pressure is not big enough to offset the high inflation expectation. Therefore, the realized inflation is positive even at a time of deep recession.

Now let us turn our attention to analyzing whether the Calvo model and the Rotemberg model produce different or similar results in the fully nonlinear framework. At this moment, it is worth reminding the reader that the parameters in the two models are calibrated such that they generate the same results at the first-order approximation. In addition, the deterministic steady state is the same in the two models.

From Figure 1, when the nominal interest is positive in the two models, these models generate a very similar policy function for consumption, net output (consumption plus government spending), hours worked, real marginal cost, and government spending.\textsuperscript{7} However, it is surprising that the policy function for the nominal interest rate, inflation, and the real interest rates are very different before the ZLB binds. Specifically, the nominal interest rate cut is larger, compared to the case where no ZLB is imposed, in the Calvo model than in

\textsuperscript{6}See Adam and Billi [2007] and Ngo [2014] for more detailed explanation.

\textsuperscript{7}We compare net output, measured as consumption plus government spending. Note that, in the Calvo model, output is the same net output, while in the Rotemberg model, output is net output plus price adjustment costs.
the Rotemberg model given the same shock. Consequently, the ZLB is more likely to bind in the Calvo model than in the Rotemberg model. In other words, for the ZLB to bind, we need a smaller preference shock in the Calvo model than in the Rotemberg model.

It is also interesting that when the ZLB binds in the two models, the Calvo model generates a more severe recession. Given the same positive magnitude of preference shock that drives the ZLB to bind in the two models, the output loss and inflation decline are larger in the Calvo model than in the Rotemberg model, as seen in Panels B and E. These results come from the fact that the real interest rate, as seen in Panel C of Figure 1, is much higher in the Calvo model than in the Rotemberg model, leading to more incentive for households to save in the Calvo model, resulting in a sharper decline in consumption and output in the Calvo model than in the Rotemberg counterpart.

The question is why the two models, though equivalent at the first-order approximation, generate such different results in the fully nonlinear framework, especially when the ZLB binds. The answer lies in the relative price dispersion variable and its role as a negative technology shock. Note that in the Calvo model, the relative price dispersion is $\Delta_t$. This state variable equals one at the first-order approximation around the steady state with zero inflation, so it has no impact on the other variables regardless of the magnitude of shocks and inflation. In the Rotemberg model, the relative price dispersion is always equal to one because all firms charge the same price given the fact that they are endowed with the same initial price, $P_0$. This is the reason the two models are equivalent at the first-order approximation.

However, this is not the case in the fully nonlinear framework. In this framework, while the relative price dispersion always equals one in the Rotemberg model, it is no longer one in the Calvo model. Under a big shock driving far away from the steady state, the relative price dispersion rises. Especially, when the ZLB binds and the central bank is not able to obtain the zero target inflation rate, the state variable is significantly greater than one and
is the main factor that causes the difference between the Calvo and Rotemberg models.

As in Equation (25), the higher the relative price dispersion, the more inefficient the economy. The relative price dispersion variable plays the role of a negative technology shock. Therefore, we see more output decrease in the Calvo model than in the Rotemberg model when the role of the relative price dispersion kicks in. Thus, the central bank in the Calvo model has more incentive in cutting back the nominal interest rate to obtain the zero target inflation, and, as a result, a smaller relative price dispersion. Hence, the probability for the nominal interest rate to hit the ZLB is larger in the Calvo model than in the Rotemberg model.

However, when the ZLB binds, the central bank is not able to obtain inflation that close to the target. The decline in output results in a sharp decline in inflation and a rise in the relative price dispersion, as seen in Panel H of Figure 1. This increase in the relative price dispersion, in turn, causes output and inflation fall more, and so on. Therefore, eventually, output and inflation fall more in the Calvo model than in the Rotemberg model, mainly due to the effect of the increase in the relative price dispersion.

### 4.2 Long run distribution of inflation and interest rates

To compute the unconditional probability of hitting the ZLB, we first solve the two models. Then, based on the policy function, we generate simulated series of interest and inflation rates for 300,000 periods starting from the initial steady state. Based on the simulated series, we draw the unconditional distribution of the nominal interest rate and the inflation rate with respect to shocks. We also compute the unconditional probability of hitting the ZLB and the long-run inflation.

Using the benchmark calibration, we find that the unconditional probability of hitting the ZLB is 5.6% in the Calvo model, while it is only 3.00% in the Rotemberg model. As explained above, due to the impact of the relative price dispersion, the nominal interest rate
is cut more aggressively in the Calvo model than in the Rotemberg model, resulting in a higher unconditional and conditional probability of hitting the ZLB in the Calvo model.\footnote{We also computed the probability of hitting the ZLB conditional on state of preference, the duration of staying at the ZLB conditional on how long the economy has stayed at the ZLB. The results can be available upon requests.}

It is not surprising that the long-run inflation rates are negative and approximately zero in both models. The deflation bias is 0.1\% in the Calvo model, slightly bigger than 0.01\% in the Rotemberg model. The intuition is that the higher the probability of hitting the ZLB, the higher the risk of falling into the liquidity trap with deflation. Therefore, the unconditional inflation is less than zero given the fact that the deterministic steady state inflation rate is zero.

\subsection*{4.3 Government spending multipliers}

The effectiveness of fiscal policy receives much attention from economists and policymakers recently when the target federal funds rate hits the ZLB and the conventional monetary policy is not effective in stimulating economic activities. Woodford [2011] argues that the government spending multiplier is possibly well in excess of one when monetary policy is constrained by the ZLB, especially in the case where government purchases expand to partially fill the output gap that arises from the inability to lower the nominal interest rate. Christiano et al. [2011] also show that the government expenditure multiplier is far above one when the economy is stuck in the liquidity trap with a binding ZLB.

In this section, we compute and report two types of government spending multipliers: the impact multiplier and the cumulative multiplier. In addition, we report the multipliers in two cases: when the economy is inside the ZLB and when it is outside the ZLB.
4.3.1 Government spending multipliers when the ZLB is not binding

To compute the multipliers when the economy is outside the ZLB, we first assume that initially in period 0, the economy stays at the steady state. We then impose a shock to the share of government spending, $g$, in period 1 such that the government spending, $G$, increases by 1% from the steady state value.\(^9\) The share of government spending is allowed to return to the steady state based on its motion equation. Using the approximated policy function, we compute and keep track the path of output and government spending $(Y_t, G_t)_{t=0}^T$, where $T$ is big enough to make sure that the economy will converge to the steady state.

The impact multiplier and the cumulative multiplier are computed as follows:

$$m^{NoZLB}_{impact} = \frac{Y_1 - Y_0}{G_1 - G_0}$$

$$m^{NoZLB}_{cumulative} = \frac{\sum_{t=1}^{T} (Y_t - Y_0)}{\sum_{t=1}^{T} (G_t - G_0)}$$

where $G_0$ and $Y_0$ are the steady state government spending and the steady state output.

We are also interested in seeing how government spending persistence affects government spending multipliers in the two models. To this end, we solve the models with different values of the government spending shock persistence, $\rho_g$, then compute the impact and cumulative multipliers when the economy is outside the ZLB as described above. The results are presented in Figure 2. Note that, in this case, the impact multiplier is similar to the cumulative multiplier. To save the space, we do not show a separate graph for the impact multiplier.

In this case of no binding ZLB, the two models produce very similar government spending multipliers at different levels of the persistence of government spending shock. At the benchmark parameterization where the persistence of the government spending shock is 0.8, the results in this case are quite robust to larger sizes of the government spending shock, such as 3% or 5%.

\(^9\)The results in this case are quite robust to larger sizes of the government spending shock, such as 3% or 5%. 

22
Persistence of the government spending shock, $\rho_g$

Figure 2: Cumulative government spending multipliers when the ZLB is not binding. The government spending shock is 1%, and the preference shock is 0%. Note that in this case, the impact multiplier is very similar to the cumulative multiplier.
multiplier is around 0.75 in both models. As explained above, in this case, the central bank can obtain output and inflation close to the target; therefore, the relative price dispersion is insignificantly different from one in the Calvo model and has no further significant impact on inflation and output.

There are two interesting features of Figure 2. First, the multiplier is always smaller than one in both models no matter how persistent the government spending shock is. This occurs because of the well-known crowding-out effect of government spending on private consumption. Specifically, an increase in government spending puts upward pressure on output and inflation. The central bank will raise the nominal interest rate through the Taylor rule such that the real interest increases to fight against higher inflation. Consequently, private consumption falls, and, as a result, the increase in output is smaller than the increase in government spending. Eventually, the government spending multiplier is less than one.

Second, the multiplier decreases in the persistence of the government spending shock. The intuition is that, given the initial 1% increase in government spending, the more persistent the government spending shock, the larger government spending is and, as a result, the larger inflation is in the future. Forward-looking firms will increase their current prices if they have a chance to do so, resulting in a larger current inflation. According to the Taylor rule, the central bank will raise the nominal interest rate and the real interest rate, leading to a larger crowding-out effect on private consumption. Eventually, the impact and cumulative multipliers are smaller as the persistence increases.

4.3.2 Government spending multipliers when the ZLB is binding

This case is of particular interest because of the call for fiscal policy expansion during and after the Great Recession, where the target federal fund rate has been at the ZLB since December 2008. To compute the impact multiplier and cumulative multiplier when the economy is at the ZLB, we follow the steps described below.
First, we assume that the economy stands at the steady state initially in period $t = 0$. We impose a positive preference shock in period 1 such that, without any other shocks, the economy will stay at the ZLB for five periods. The preference shock is allowed to return to the steady state based on its motion equation. We compute and keep track of the path of output and government spending under the only preference shock, $(Y^1_t, G^1_t)_{t=0}^T$.

Second, on top of the preference shock, in period 1, we impose a shock to the share of government spending, $g$, such that the government spending, $G$, in period 1 increases by 1% from the value generated under only the preference shock. Both the preference and government spending shocks are allowed to return to the steady state based on their motion equations. We compute and keep track of the path of output and government spending under the two shocks, $(Y^2_t, G^2_t)_{t=0}^T$. The number of periods for the exercise, $T$, is big enough to make sure that the economy will converge to the steady state.

The impact multiplier and the cumulative multiplier when the economy is at the ZLB are computed as follows:

$$m_{\text{impact}}^{ZLB} = \frac{Y_1^2 - Y_1^1}{G_1^2 - G_1^1}$$
$$m_{\text{cumulative}}^{ZLB} = \frac{\sum_{t=1}^{T} (Y_t^2 - Y_t^1)}{\sum_{t=1}^{T} (G_t^2 - G_t^1)}$$

As in the case where the ZLB is not binding, in this case we compute the impact and cumulative multipliers for different values of the government spending persistence. The impact multiplier at different values of persistence is presented in Figure 3, while the cumulative multiplier is shown in Figure 4.10

At the benchmark calibration where the persistence is 0.8, the impact multiplier is around 1.15 in the Calvo model, while it is around 0.95 in the Rotemberg model. These values are much smaller than those reported in the existing literature. Woodford [2011] and Christiano

---

10The results, both magnitudes and patterns of the multipliers, in this case are quite robust to larger sizes of the government spending shocks, such as 3% or 5%.
Persistence of the government spending shock, $\rho_g$

Calvo model
Rotemberg model

Figure 3: Impact government spending multipliers when the ZLB is binding due to an adverse preference shock. The government spending shock is 1%, and the preference shock is 0.9%.

et al. [2011] find that the impact multiplier is around 2 in their benchmark model. The main reason for the discrepancy is that the shock driving the nominal interest to the ZLB is much more persistent in their models than in this model. Specifically, the persistence of the shock to the natural rate of interest rate is calibrated to be 0.903 in Woodford’s paper, while it is 0.8 persistence of the preference shock in this paper. Hence, the ZLB in his model binds approximately 10 quarters on average instead of 5 quarters as in this paper. We will investigate how the duration of the ZLB affects the multipliers in the next section.

From Figure 3, it is apparent that the two models produce very different government spending multipliers on impact. First, while the impact multiplier is always smaller than one in the Rotemberg model regardless of the magnitude of the government spending persistence, it can be larger than one in the Calvo model. Second, the multiplier in the Calvo model first increases then decreases in the persistence, while the pattern is not clear in the Rotemberg
Figure 4: Cumulative government spending multipliers when the ZLB is binding due to an adverse preference shock. The government spending shock is 1%, and the preference shock is 0.9% per quarter.
model. Third, on impact, the multiplier is always greater in the Calvo model than in the Rotemberg model.

The less-than-one impact multiplier in the Rotemberg model, which is robust to the persistence of the government spending shock, is surprising because there is a conventional belief that when the ZLB binds, the government spending multiplier is greater than one in any New Keynesian model with sticky prices, as discussed in Christiano et al. [2011], Woodford [2011], Eggertsson and Krugman [2012], and many others. It is worth noting that the pricing behaviors in the existing literature are of the Calvo type.

The question then is why the multiplier is greater in the Calvo model than in the Rotemberg model when the ZLB is binding. As explained above in the the policy function section, the relative price dispersion can be significantly greater than one - the steady state value - in the Calvo model if the ZLB binds, while the dispersion is always one in the Rotemberg model. The government spending in the Calvo model has two effects. First, as in the Rotemberg model, government spending generates inflation directly, leading to a lower real interest rate given the nominal interest rate stuck at the ZLB, resulting in higher output and inflation that are close to the target. The closer the inflation to the target, the smaller the relative price dispersion, leading to a smaller output loss and inflation closer to the target, and so on. Therefore, the impact of government spending is much stronger in the Calvo model than in the Rotemberg model due to the effect of the relative price dispersion.

It is important to note that in both models, though it is not very clear in the Rotemberg model, the impact multiplier is nonmonotonically related to the persistence of the government shock. It first increases then decreases in the persistence. When the persistence of the government increases, higher future inflation is created, leading to a higher current inflation rate and, as a result, a smaller real interest rate as long as the ZLB binds. That is why the impact multiplier first increases in the persistence of the government spending. However, if the shock is too persistent, the inflation generated is too big, causing a rise in the nominal
interest rate from the ZLB, and, as a result, a higher real interest rate compared to the case of lower persistence. This causes the well-known crowding-out effect on private inflation. Therefore, the impact multiplier starts decreasing when the persistence is greater than a certain level.

The same features occur with the cumulative government spending multiplier, as seen in Figure 4. The cumulative multiplier is always smaller than one in the Rotemberg model, but it can be larger than one in the Calvo model if the government spending shock is not very persistent. In both models, the cumulative multiplier first increases then decreases in the persistence of the government spending shock.

The nonmonotonic relationship between the multipliers and the persistence of government spending shock in the Calvo model is novel. Woodford [2011] finds that the relationship is monotonic. Specifically, the multiplier monotonically decreases in the persistence, as seen in Figure 3 of his paper. The reason for the difference lies in the fact that Woodford [2011] uses the linearized version of the full nonlinear Calvo model presented in this paper. Hence the role of the relative price dispersion almost disappears, as in the Rotemberg model where we do not see the nonmonotonic relationship clearly.

It is also worth noting that the maximal cumulative multiplier in the Calvo model is far less than 1.05, while the maximal impact multiplier in the Calvo model is around 1.18. In addition, compared with the impact multiplier, the cumulative multiplier starts falling sooner when the persistence of government spending shock keeps rising.

The intuition for why the maximal cumulative multiplier is less than the maximal impact multiplier is that: when the government spending shock is more persistent, it is likely that some part of government spending expansion occurs when the ZLB is not binding. Note that without the government spending shocks, the ZLB binds five periods; beyond this time the ZLB is no longer binding and the government multiplier starts falling. Hence, the maximal cumulative multiplier is smaller than the maximal impact multiplier.
Figure 5: Cumulative government spending multipliers when the ZLB is binding. The government spending shock is 1%, and the preference shock is 0.9% when the ZLB binds.

5 Sensitivity analysis

In the previous section, we have done several analyses regarding the sensitivity of the government spending multiplier with respect to the persistence of the government spending shock. We find that the relationship between the multiplier and the persistence is nonmonotonic and that the multipliers are always less than one in the Rotemberg model regardless of the magnitude of the persistence.

In this section, we would like to answer the question: how does the duration of being at the ZLB affect the multipliers? Both Woodford [2011] and Christiano et al. [2011] find that the longer the economy stays at the ZLB, the larger the government spending multiplier, and the multiplier can be unboundedly large if the duration is very long.

To answer this question, we first impose a path of preference shock such that without other shocks to government spending, the economy will stay at the ZLB for $T_{ZLB} \geq 5$
periods, instead of five periods as in the benchmark exercise. Whenever the economy stays at the ZLB, net output is around 3% smaller than the steady state in the Calvo model. We then compute the cumulative government spending multipliers when the economy is at the ZLB, as described above. The results are presented in Figure 5.

It is not surprising that when the duration of the binding ZLB increases, the cumulative multiplier increases in both models. However, it is interesting that the multiplier converges to around 1.17 in the Calvo model and around to 0.95 in the Rotemberg model. These values are much smaller than those reported in the existing literature, including Woodford [2011], Christiano et al. [2011], and Eggertsson and Krugman [2012], where they find that the multiplier is around 2.

6 Conclusion

The paper explores the difference between the Calvo model and the Rotemberg counterpart in a fully nonlinear DSGE framework with occasionally binding ZLB. Although the two models are equivalent at the first-order approximation, they generate very different results regarding the probability of hitting the ZLB, the long-run inflation, the impact and cumulative government spending multipliers, and the sensitivity of the multipliers with respect to the persistence of the government shock and the preference shock that drive the ZLB to bind.

The most notable difference between the two models is that, while the impact and cumulative multipliers are less than one in the Rotemberg model when the ZLB binds, they can be larger than one in the Calvo model. The underlying reason is that the relative price dispersion plays a very important role in the Calvo model, especially when the central bank cannot stabilize inflation and output. The dispersion does not exist in the Rotemberg model.

The less-than-one government spending multipliers challenge the widely held belief that
all New Keynesian models with some sticky prices would produce a greater-than-one government spending multiplier when the nominal interest rate is held constant at the ZLB. The findings in this paper also suggest that there might be other classes of New Keynesian models that would generate less-than-one government spending multipliers even when the ZLB binds. For example, it would be interesting to see how large the government spending multipliers generated by the class of state-dependent-pricing models, such as Dotsey et al. [1999], are instead of the time-dependent pricing model, such as the Calvo model that is commonly used in the existing literature.

7 References

References


8 Appendix

8.1 Appendix A: Models

8.1.1 A model with Calvo price adjustments

\[(1) \quad \frac{1+i_t}{1+i} = \max \left\{ \frac{1}{1+i}, \left( \frac{Y_t}{\bar{Y}} \right)^{\phi_y} \left( \frac{1+\pi_t}{1+\pi} \right)^{\phi_{\pi}} \right\} \]

\[(2) \quad \frac{C_t^{1-\gamma}}{1+i} = \beta \beta_t E_t \left( \frac{C_{t+1}^{1-\gamma}}{1+\pi_{t+1}} \right) \]

\[(3) \quad F_t = C_t^{1-\gamma} Y_t + \theta \beta \beta_t E_t \left( (1+\pi_{t+1})^{\varepsilon-1} F_{t+1} \right) \]
\begin{align*}
(4) : S_t &= \frac{\chi Y_t N_t^\eta}{(1 - \Phi)} A_t + \theta \beta_t E_t ((1 + \pi_{t+1})^\varepsilon S_{t+1}) \\
(5) : \frac{S_t}{F_t} &= \left(\frac{1 - \theta (1 + \pi_t)^{\varepsilon-1}}{1 - \theta}\right)^{\frac{1}{\varepsilon-1}} \\
(6) : Y_t &= C_t + G_t \\
(7) : Y_t &= N_t A_t (\Delta_t)^{-1} \\
(8) : \Delta_t &= (1 - \theta) \left(\frac{1 - \theta (1 + \pi_t)^{\varepsilon-1}}{1 - \theta}\right)^{\frac{\varepsilon}{\varepsilon-1}} + \theta (1 + \pi_t)^{\varepsilon} \Delta_{t-1}
\end{align*}

where
\[\Phi = 1 - (1 - \varepsilon^{-1});\]

\(\varepsilon\) is the elasticity of substitution among differentiated goods; \(\theta\) is the probability of keeping a price unchanged; \(G\) denotes the government expenditure:

\[G_t = Y_t S_g \exp (g_t)\]

or

\[G_t = Y S_g \exp (g_t)\]

and \(\beta_t, g_t, \text{ and } A_t\) are preference shocks, government spending shocks, and technology shocks respectively. Specifically,

\[
\ln (\beta_{t+1}) = \rho_\beta \ln (\beta_t) + \varepsilon_{\beta,t+1} \\
\ln (g_{t+1}) = \rho_g \ln (g_t) + \varepsilon_{g,t+1} \\
\ln (A_{t+1}) = \rho_A \ln (A_t) + \varepsilon_{A,t+1}
\]

where \(\beta_s \in (0, 1)\) is the persistence of the shock \(s \in \{\beta, g, A\}\) and \(\varepsilon_{s,t}\) is i.i.d \((0, \sigma^2_s)\).
8.1.2 A model with Rotemberg price adjustments

\[ (1) : \frac{1 + i_t}{1 + i} = \max \left\{ \frac{1}{1 + i^t} \left( \frac{Y_t}{Y} \right)^{\phi_y} \left( \frac{1 + \pi_t}{1 + \pi} \right)^{\phi_{\pi}} \right\} \]

\[ (2) : \frac{C_t^{-\gamma}}{1 + i_t} = \beta \beta_t E_t \left( \frac{C_{t+1}^{-\gamma}}{1 + \pi_{t+1}} \right) \]

\[ (3) : \left( 1 - \varepsilon + \varepsilon \frac{w_t}{A_t} - \varphi \pi_t (1 + \pi_t) \right) Y_t^g C_t^{-\gamma} + \varphi \beta \beta_t E_t \left[ C_{t+1}^{-\gamma} \pi_{t+1} (1 + \pi_{t+1}) Y_{t+1}^g \right] = 0 \]

\[ (4a) : Y_t = C_t + G_t \]
\[ (4b) : Y^g_t = Y_t + \frac{\varphi}{2} \pi^2_t Y_t^g \]

\[ (5) : Y^g_t = A_t N_t \]

\[ (6) : \frac{\chi N^\eta_t}{C_t^{-\gamma}} = w_t \]

where \( \varphi \), the adjustment cost parameter, presents how costly it is for firms to change their prices; \( Y^g \) is the output plus adjustment cost.

8.2 Appendix B: The New-Keynesian IS and Phillips Curve

From the production function, the marginal trade-off between consumption and labor, and the aggregate equation in the Calvo model:

\[ \hat{Y}_t = \hat{N}_t + A_t \]
\[ \eta \hat{N}_t + \gamma \hat{C}_t = \hat{w}_t \]
\[ \hat{Y}_t = (1 - S_g) \hat{C}_t + S_g g_t \text{ or } \hat{Y}_t = \hat{C}_t + \frac{S_g}{1 - S_g} g_t \]
With flexible prices, \( \hat{w}_t = A_t \) because the marginal cost is constant, we have:

\[
\hat{Y}_f^t = N^f_t + A_t \\
\eta \hat{N}_t^f + \gamma \hat{C}_t^f = A_t \\
\hat{Y}_f^t = (1 - S_g) \hat{C}_t^f + S_g g_t \text{ or } \hat{Y}_f^t = \hat{C}_t^f + \frac{S_g}{1 - S_g} g_t
\]

Then we can find:

\[
\hat{C}_t - \hat{C}_t^f = \frac{1}{1 - S_g} x_t \text{ or } \hat{C}_t - \hat{C}_t^f = x_t \\
\hat{C}_t^f = \frac{1 + \eta}{\eta (1 - S_g) + \gamma} A_t - \frac{\eta S_g}{\eta (1 - S_g) + \gamma} g_t \text{ or } \hat{C}_t^f = \frac{1 + \eta}{\eta + \gamma} A_t - \left( \frac{\eta}{\eta + \gamma} \right) \left( \frac{S_g}{1 - S_g} \right) g_t \\
w_t - A_t = (\eta + \gamma) x_t
\]

From the Euler equation:

\[
\hat{C}_t = E_t \hat{C}_{t+1} - \frac{1}{\gamma} (i_t - E_t \pi_{t+1}) - \frac{1}{\gamma} \beta_t \\
\hat{C}_t - \hat{C}_t^f = E_t \left( \hat{C}_{t+1} - \hat{C}_{t+1}^f \right) + E_t \left( \hat{C}_{t+1}^f - \hat{C}_t^f \right) - \frac{1}{\gamma} (i_t - E_t \pi_{t+1}) - \frac{1}{\gamma} \beta_t \\
x_t = E_t x_{t+1} - \frac{1}{\gamma} (i_t - E_t \pi_{t+1}) + \frac{1}{\gamma} r^n_t
\]

where

\[
r^n_t = -\beta_t + \frac{\gamma (1 + \eta) (1 - S_g)}{\eta (1 - S_g) + \gamma} (\rho_A - 1) A_t - \frac{\gamma S_g (1 - S_g)}{\eta (1 - S_g) + \gamma} (\rho_G - 1) g_t \\
or r^n_t = -\beta_t + \frac{\gamma (1 + \eta)}{\eta + \gamma} (\rho_A - 1) A_t - \left( \frac{\gamma \eta}{\eta + \gamma} \right) \left( \frac{S_g}{\eta + \gamma} \right) (\rho_G - 1) g_t
\]

At the first-order approximation around the steady state with zero inflation, the optimal
pricing rule in the Calvo model becomes:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t$$

and the optimal pricing equation in the Rotemberg model becomes:

$$\pi_t = \beta E_t \pi_{t+1} + \tilde{\kappa} x_t$$

where

$$\kappa = \left( \eta + \gamma \right) \left( 1 - \theta \right) \left( 1 - \theta \beta \right) / \theta$$

$$\tilde{\kappa} = \left( \eta + \gamma \right) (\varepsilon - 1) / \varphi$$

### 8.3 Appendix C: Solution method

This appendix shows the solution method used to solve the Calvo model. The Rotemberg model can be solved in the same way. Follow Miranda and Fackler [2002], we rewrite the functional equations governing the equilibrium in the Calvo model in a more compact form:

$$f(s, X(s), E[Z(X(s'))]) = 0$$

Here $f : R^{3+7+3} \rightarrow R^7$ is the equilibrium relationship.

where

$s = (\Delta, \beta, g)$ is the current state of the economy

$X(s) = (R(s), C(s), N(s), S(s), F(s), \Pi(s), Y(s))'$ is the policy function we need to solve,

$X : R^3 \rightarrow R^8$.

$s'$ is next period state that evolves according to the following motion equation:
\[ s' = g(s, X(s), \varepsilon) = \begin{bmatrix}
\Delta' = (1 - \theta) \left( \frac{-\theta \Pi(s)^{\varepsilon-1}}{1-\theta} \right)^\frac{1}{\varepsilon-1} + \theta \Pi(s)^{\varepsilon} \Delta \\
\beta' = \beta^o \exp(\varepsilon_{\beta}) \\
g' = g^o \exp(\varepsilon_g)
\end{bmatrix} \]

\( \varepsilon_{\beta} \) and \( \varepsilon_g \) are the innovations of the preference and the government spending shocks.

\[ Z(X(s')) = \begin{bmatrix}
Z_1(X(s')) = \frac{C(s')^{-\gamma}}{\Pi(s')} \\
Z_2(X(s')) = \Pi(s')^{\varepsilon-1} F(s') \\
Z_3(X(s')) = \Pi(s')^{\varepsilon} S(s')
\end{bmatrix} \]

We solve the above equilibrium relationship using a projection method called the collocation method. Below is the simplified algorithm:

**Step 1:** Define the space of the approximating functions and collocation nodes 
\( S = (S_1, ..., S_N) \), where \( N = N_\Delta \times N_\beta \times N_g \) and \( (N_\Delta \times N_\beta \times N_g) \) is the polynomial degree in each dimension of the space. In this paper, we use the cubic spline method where \( N_\Delta \times N_\beta \times N_g \) are the number of collocations nodes along each state dimension.

\[ X(s) = (\phi(s)\theta_R, \phi(s)\theta_C, \phi(s)\theta_N, \phi(s)\theta_F, \phi(s)\theta_S, \phi(s)\theta_\Pi, \phi(s)\theta_\Delta, \phi(s)\theta_Y)' \]

or \( X(s) = \phi(s)\Theta \)

where

- \( \phi(s) \) is a \( 1 \times N \) matrix of cubic spline basis functions evaluated at state \( s \in S = (S_1, ..., S_N) \).
- \( \Theta = (\theta_R; \theta_C; \theta_N; \theta_F; \theta_S; \theta_\Pi; \theta_\Delta; \theta_Y) \) is \( N \times 13 \) coefficient matrix that we want to approximate.

**Step 2:** Initialize the coefficient matrix \( \Theta^0 \), and set up stopping rules.

**Step 3:** At each iteration \( j \) we have a corresponding \( \Theta^j \), implement the following substep:

1. At each collocation node \( s_i, s_i \in \{S_1..S_N\} \) : compute \( E[Z(X(s'))] \) :
2. Solve for \( X(s_i) \) s.t. \( f(s_i, X(s_i), E[Z(X(s'))]) = 0 \). We solve this complementarity problem using the Newton method with a user "analytical" Jacobian matrix \( f_X \).

**Step 4:** Update \( \Theta^{j+1} = \Phi^{-1}\Theta^j \), where \( \Phi = (\phi(s_1), ..., \phi(s_N))' \).

**Step 5:** Check the stopping rules, if not satisfied go to Step 3, otherwise go to Step 6.

**Step 6:** Report results.

There is another way to solve for the policy functions. We can define the residual function, \( r(s, \Theta) = f(s, X(s), E[Z(X(s'))]) \) and use Newton’s method to solve \( r(s, \Theta) = 0 \) by updating \( \Theta^{j+1} = \Theta^j - \alpha [r_\Theta (s, \Theta^j)]^{-1} r(s, \Theta^j) \), where \( r_\Theta (s, \Theta^j) \) is user "analytical" Jacobian matrix.

### 8.3.1 Jacobian matrix

If we use the iteration approach and the Newton method to solve the system at each state

Analytical Jacobian, or the first derivative w.r.t \( X_t = (R, C, N, S, F, \Pi, \Delta, Y)_t \):

\[
f_{1X} = \begin{cases} 
0, 0, 0, 0, 0, 0, 0, 0, \phi_x \left( \frac{\Pi}{Y} \right) \phi_y^{-1} & \text{if } \left( \frac{\Pi}{Y} \right) \phi_x \left( \frac{Y}{Y} \right) \phi_y > 1 - R_t \\
-1/R, 0, 0, 0, 0, 0, 0, 0 & \text{if } \left( \frac{\Pi}{Y} \right) \phi_x \left( \frac{Y}{Y} \right) \phi_y \leq 1 - R_t
\end{cases}
\]

\[
f_{2X} = \left( -\frac{C_t^{-\gamma}}{R_t}, -\gamma C_t^{-\gamma-1}, 0, 0, 0, 0, -\beta \beta_t E_t \left[ Z_{1\Delta t+1} \right], 0; \right)
\]

\[
f_{3X} = \left( 0, (-\gamma) C_t^{-\gamma-1} Y_t, 0, 0, 0, 0, -\theta \beta_t E_t \left[ Z_{2\Delta t+1} \right], -C_t^{-\gamma}; \right)
\]
\[ f_{4X} = \left( 0, 0, -\frac{\chi \eta Y_t N_t^\eta}{(1 - \Phi) A_t}, 0, 1, 0, -\theta \beta \beta_t E_t [Z_{3\Delta t+1}], -\frac{\chi N_t^\eta}{(1 - \Phi) A_t}; \right) \]

\[ f_{5X} = \left( 0, 0, 0, -\left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \theta} \right)^{\frac{1}{\varepsilon}}, 1, -F_t \left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \theta} \right)^{\frac{1}{\varepsilon}} \theta \Pi_t^{\varepsilon-2} \frac{1}{1 - \theta}, 0, 0; \right) \]

\[ f_{6X} = (0, 1, 0, 0, 0, 0, -1 + S_g \exp (g_t); ) \]

\[ f_{7X} = \left( 0, 0, 1, 0, 0, 0, 0, 0, \frac{N_t A_t}{\Delta^2_t}, 1; \right) \]

\[ f_{8X} = \left( 0, 0, 0, 0, 0, \left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \theta} \right)^{\frac{1}{\varepsilon}}, \varepsilon \theta \Pi_t^{\varepsilon-2} - \theta \varepsilon \Pi_t^{\varepsilon-1} \Delta_{t-1}, 1, 0; \right) \]

where

\[ Z_{1\Delta t+1} = -\frac{\gamma C_{t+1}^{\varepsilon-1} C_{\Delta t+1}}{\Pi_{t+1}} - \frac{C_{t+1}^{\varepsilon-1} \Pi_{t+1}}{\Pi_{t+1}^2} \]

\[ Z_{2\Delta t+1} = (\varepsilon - 1) \Pi_t^{\varepsilon-2} \Pi_{\Delta t+1} F_t + \Pi_t^{\varepsilon-1} F_{\Delta t+1} \]

\[ Z_{3\Delta t+1} = \varepsilon \Pi_t^{\varepsilon-1} \Pi_{\Delta t+1} S_{t+1} + \Pi_t^{\varepsilon} S_{\Delta t+1} \]