Variance Decomposition Analysis for Nonlinear DSGE Models: An Application with ZLB*

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Abstract

In this paper, we first proposes two new methods and algorithms to compute and analyze variance decomposition for nonlinear DSGE models. We then apply these methods to a standard DNK model with occasionally-binding zero lower bound on nominal interest rates (ZLB). Specifically, we study the relative importance of supply and demand shocks to business cycles before and at the ZLB. We find that the supply shock becomes significantly less important relative to the demand shock in explaining the volatility of economic variables, especially GDP, at a short horizon when the economy stays at a deep recession with binding ZLB. This occurs because the reponses of the economic variables under a technology shock become relatively smaller than those under a preference shock when the ZLB binds. We also show that our methods out-perform the only existing method proposed by Lanne and Nyberg (2016).

JEL classification: C15, C32, C61, E37, E52.
Keywords: Forecast error variance decomposition, nonlinear DSGE models, ZLB, law of total variance, Delta method, projection methods.

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1 Introduction

Fully-nonlinear methods for DSGE models such as Dynamic New-Keynesian (DNK) models have been increasingly popular recently due to the occasionally-binding zero lower bound constraint in monetary policy. Many researchers have found that the results based on fully-nonlinear method would be very different from those based on linear (or partially nonlinear) method, see Fernandez-Villaverde et al. (2015) and Ngo (2016).[1]

With linear solution methods and, as a result, linear policy function, our computation and analysis of forecast error variance decomposition are straightforward. But things become more complicated when models and policy function are fully-nonlinear because both impulse response function and variance decomposition are not only state dependent but also shock and composition dependent, see Koop et al. (1996). Although, the generalized impulse response function as proposed in Koop et al. (1996) has been used more frequently in the ZLB literature, researchers tend to ignore variance decomposition either because there has not been an easy way to implement it or because they do not think it is important. However, Gourio and Ngo (2016) shows that the ZLB constraint would alter the response of economic variables to shocks, leading to change in relative importance of shock to risk premia and business cycles.

This paper first proposes two new methods to compute and analyze forecast error variance decomposition for non-linear DSGE models. The first method is called the total variance method because it is an application of the law of total variance. This method provides the true forecast error variance decomposition (FEVD). The second method is called the Delta method because is an application of the Delta method (or Taylor expansions). While the first method generates the true FEVD, the Delta method only produces an approximation of the true FEVD. We then apply these two methods to a standard DNK model with occasionally-binding ZLB. Specifically, we study the relative importance of shocks to business cycles before and at the ZLB. We also compare our methods to the one recently proposed by Lanne and Nyberg (2016).

Our main findings include: (i) when the economy is at a deep recession with binding ZLB, the supply shock becomes significantly less important relative to the demand shock in explaining the volatility of economic variables, especially at short horizons. This occurs

[1] In addition to the papers cited in the main text, an incomplete list of papers using nonlinear models with a ZLB constraint includes Wolman (2005), Nakata (2016), Ngo (2014b), Richter and Throckmorton (2015), Miao and Ngo (2016), and Richter et al. (2014).

because the reponse of GDP to technology shock becomes much smaller relative to the response of GDP to preference shock at the Great Recession state with binding ZLB. (ii) the Delta method generates the results closer to the true values, which are produced by the total variance method, than the Lanne and Nyberg method does. Thus, the Delta method is more accurate than the Lanne and Nyberg method. In addition, the Lanne and Nyberg method totally relies on simulation, while the Delta method utilizes the partial derivatives of the policy function. Hence, the Delta method is faster than the Lanne and Nyberg method; (iii) In addition, we provide two new algorithms for our new methods.

This paper is related to the literature that investigates impulse responses and forecast error variance decomposition (FEVD) for nonlinear vector autoregression (VAR) models. While the literature that studies impulse responses for nonlinear VAR models was dated back to the seminal work by Koop et al. (1996), the research in FEVD for nonlinear models has been limited. The only paper related to ours is Lanne and Nyberg (2016), where they propose a new method to compute FEVD in nonlinear models, which is called generalized forecast error variance decomposition (GFEVD). Their method is very similar to the traditional forecast error variance decomposition for a linear VAR model. The only small modification is that they use generalized impulse responses instead of traditional impulse response in their formula. Other than that, their method does not have any clear theoretical background. Therefore, it might not be accurate. On the other hand, our newly-proposed methods are derived based on probability and statistic theories.

In terms of solution methods to compute nonlinear policy function, our paper is most closely related to the papers by Judd et al. (2011), Fernandez-Villaverde et al. (2015), Ngo (2014b), Gust et al. (2012), and Aruoba and Schorfheide (2013).[2] All these papers use global projection methods to approximate agents' decision rules in a DNK model with a ZLB constraint.

The paper is organized as follows. The new methods to compute FEVD for a nonlinear DSGE model are introduced in Section 2. We also summarize the GFEVD proposed by Lanne and Nyberg (2016) in this section. In Section 3, we apply these methods to a standard DNK model with occasionally binding ZLB to study the relative importance of supply and demand shocks in explaining business cycles. In this section we also propose

[2]In addition to the papers cited in the main text, an incomplete list of papers using nonlinear models with a ZLB constraint includes Wolman (2005), Nakata (2016), Ngo (2016), Richter and Throckmorton (2015), Miao and Ngo (2016), and Richter et al. (2014).

new algorithms for our new methods. We conclude in Section 4.

2 New variance decomposition methods for DSGE models: Theoretical framework

The system of nonlinear equations governing equilibria for any DSGE model can be solved, and the resulting nonlinear policy function can be cast in a state space model as following:

$$\mathbf{Y}_t = \mathbf{f}\left(\mathbf{S}_{t-1}, \boldsymbol{\varepsilon}_t\right), \tag{1}$$

where $\mathbf{f}(\cdot) : R^{n+2k} \longrightarrow R^n$ is a known nonlinear function; $\mathbf{Y}_t$ is an $n \times 1$ vector of endogenous variables; $\mathbf{S}_{t-1} = (\mathbf{Y}_{t-1}; \mathbf{s}_{t-1})$ is the vector of state variables; $\mathbf{Y}_{t-1}$ is a $n \times 1$ vector of endogenous state variables; $\mathbf{s}_t$ is a $k \times 1$ vector of exogenous state variables that has the following motion equation:

$$\mathbf{s}_t = \mathbf{A}\mathbf{s}_{t-1} + \boldsymbol{\varepsilon}_t; \tag{2}$$

$\mathbf{A}$ is a known $k \times k$ matrix; $\boldsymbol{\varepsilon}_t$ is a $k \times 1$ vector of orthorgonal shocks with a known diagonal variance-covariance matrix Σ_ε and mean $\mathbf{0}_{k\times1}$;

This type of nonlinear policy function can be cast in the form of a nonlinear vector autoregression (VAR) model, as in Koop et al. (1996) and Lanne and Nyberg (2016). The impulse response function (IRF) based on this nonlinear policy function are shock, history, and composition dependent. It means that the response of Y_{t+h} under any *single* shock ε_t^j for $j \in \{1, .., k\}$ may depend on the state of the economy at time t, the size and sign of ε_t^j, and the sign and size of all the shocks from time t to $t+h$, $\{\boldsymbol{\varepsilon}_{t+l}\}_{l=0}^{h}$.[3] Therefore, Koop et al. (1996) propose that we use generalized impulse response function (GIRF), instead of traditional impulse response function.

In addition, due to the shock, history, and composition dependence, we cannot compute forecast error variance decomposition (FEVD) using the traditional way for a linear VAR model. Lanne and Nyberg (2016) propose a method called generalized forecast error variance decomposition (GFEVD) to implement this task. However, this method does not have a clear theoretical background. It might not be accurate relative to some other

[3]Throughout this paper, we use X to denote a component of vector $\mathbf{X}$. For example, ε denotes a single shock, which is a component of the vector of orthorgonal shock $\boldsymbol{\varepsilon}$.

ways. In the following subsections we propose two new methods to decompose forecast error variance. We also compare these methods to the GFEVD method proposed by Lanne and Nyberg (2016).

2.1 New method 1: Law of total variance

As its name indicates, the first new method is an application of the law of total variance. In particular, let Y denote a scalar component of vector variable $\mathbf{Y}$ and let $Y_{t+h}|\mathbf{S}_t, \left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}$ denote variable Y at time $t+h$ conditional on state $\mathbf{S}_t$ and a certain path of $\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}$, for $i,j \in \{1,..,k\}$. In addition, let $E\left(Y_{t+h}|\mathbf{S}_t, \left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right)$ and $Var\left(Y_{t+h}|\mathbf{S}_t, \left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right)$ denote the expectation and the variance of Y_{t+h} attributed to shock j conditional on $\mathbf{S}_t$ and $\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}$, where the expectation comes from the randomness of the future path for shock j, $\left\{\varepsilon_{t+l}^{j}\right\}_{l=1}^{h}$.

We first apply the law of total variance to compute the variance of Y_{t+h} conditional on state $\mathbf{S}_t$:

$$\begin{aligned} Var\left(Y_{t+h}|\mathbf{S}_t\right) &= E\left(Var\left(Y_{t+h}|\mathbf{S}_t, \left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right)\right) \\ &\quad + Var\left(E\left(Y_{t+h}|\mathbf{S}_t, \left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=0}^{h}\right)\right). \end{aligned} \tag{3}$$

The variance of Y_{t+h} conditional on state $\mathbf{S}_t$ contributed by shock j is the first component of the right hand side of equation (3):

$$Var^{j}\left(Y_{t+h}|\mathbf{S}_t\right) = E\left(Var\left(Y_{t+h}|\mathbf{S}_t, \left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right)\right), \tag{4}$$

where the expectation is taken over the distribution of $\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}$ and the variance is taken over the distribution of $\left\{\varepsilon_{t+l}^{j}\right\}_{l=1}^{h}$.

We then compute forecast error variance decomposition (FEVD) for shock j at horizon h conditional on state $\mathbf{S}_t$:

$$\lambda^{j}\left(h|\mathbf{S}_t\right) = \frac{Var^{j}\left(Y_{t+h}|\mathbf{S}_t\right)}{Var\left(Y_{t+h}|\mathbf{S}_t\right)}. \tag{5}$$

However the sum of FEVD might not be equal to 1. We need to modify equation (5) as below:

$$\widetilde{\lambda}^{j}\left(h|\mathbf{S}_t\right)=\frac{\lambda^{j}\left(h|\mathbf{S}_t\right)}{\sum_{i=1}^{m}\lambda^{i}\left(h|\mathbf{S}_t\right)}. \tag{6}$$

2.2 New method 2: Delta method

In this subsection, we propose the so-called Delta method to compute and analyze variance decomposition. This method is an application of the Delta method (or Taylor expansions). Again, let Y denote a scalar component of vector variable $\mathbf{Y}$, and $Y=f\left(\mathbf{Y}_{t+h-1},\mathbf{s}_{t+h-1},\boldsymbol{\varepsilon}_{t+h}\right)$ is the scalar nonlinear policy function. Applying the first-order Taylor expansions to this non-linear policy function, we obtain:

$$\begin{aligned}
Y_{t+h} \simeq\; & f\left(E\left(\mathbf{Y}_{t+h-1}|\mathbf{S}_t\right),E\left(\mathbf{s}_{t+h-1}|\mathbf{S}_t\right),E\left(\boldsymbol{\varepsilon}_{t+h}|\mathbf{S}_t\right)\right)\\
&+f_{\mathbf{Y}}^{'}\left(E\left(\mathbf{Y}_{t+h-1}|\mathbf{S}_t\right),E\left(\mathbf{s}_{t+h-1}|\mathbf{S}_t\right),E\left(\boldsymbol{\varepsilon}_{t+h}|\mathbf{S}_t\right)\right)\left(\mathbf{Y}_{t+h-1}-E\left(\mathbf{Y}_{t+h-1}|\mathbf{S}_t\right)\right)\\
&+f_{\mathbf{s}}^{'}\left(E\left(\mathbf{Y}_{t+h-1}|\mathbf{S}_t\right),E\left(\mathbf{s}_{t+h-1}|\mathbf{S}_t\right),E\left(\boldsymbol{\varepsilon}_{t+h}|\mathbf{S}_t\right)\right)\left(\mathbf{s}_{t+h-1}-E\left(\mathbf{s}_{t+h-1}|\mathbf{S}_t\right)\right)\\
&+f_{\boldsymbol{\varepsilon}}^{'}\left(E\left(\mathbf{Y}_{t+h-1}|\mathbf{S}_t\right),E\left(\mathbf{s}_{t+h-1}|\mathbf{S}_t\right),E\left(\boldsymbol{\varepsilon}_{t+h}|\mathbf{S}_t\right)\right)\left(\boldsymbol{\varepsilon}_{t+h}-E\left(\boldsymbol{\varepsilon}_{t+h}|\mathbf{S}_t\right)\right),
\end{aligned} \tag{7}$$

where $f_{\mathbf{Y}}^{'}$, $f_{\mathbf{s}}^{'}$, and $f_{\boldsymbol{\varepsilon}}^{'}$ denote the gradients with respect to the endogenous state variables $\mathbf{Y}$, the exogenous state variables $\mathbf{s}$, and the orthorgonal shocks $\boldsymbol{\varepsilon}$, relatively.

Note that we approximate Y_{t+h} around the conditional expectations of future endogenous, exogenous states, and shocks: $E\left(\mathbf{Y}_{t+h-1}|\mathbf{S}_t\right)$, $E\left(\mathbf{s}_{t+h-1}|\mathbf{S}_t\right)$, and $E\left(\boldsymbol{\varepsilon}_{t+h}|\mathbf{S}_t\right)$. We can use the policy function (1) and the motion equation (2) to trace the endogenous and the exogenous state variables, and to compute their expectations and variance-covariance.

Take the conditional variance of both sides of equation (7), we obtain:

$$\begin{aligned}
Var\left(Y_{t+h}|\mathbf{S}_t\right) \simeq\; & f_{\mathbf{Y}}^{'}\left(E\left(\mathbf{Y}_{t+h-1}|\mathbf{S}_t\right),\mathbf{A}^{h-1}\mathbf{s}_t,\mathbf{0}\right)Var\left(\mathbf{Y}_{t+h-1}|\mathbf{S}_t\right)f_{\mathbf{Y}}\left(E\left(Y_{t+h-1}|\mathbf{S}_t\right),\mathbf{A}^{h-1}\mathbf{s}_t,\mathbf{0}_{k\times 1}\right)\\
&+f_{\mathbf{s}}^{'}\left(E\left(\mathbf{Y}_{t+h-1}|\mathbf{S}_t\right),\mathbf{A}^{h-1}\mathbf{s}_t,\mathbf{0}\right)Var\left(\mathbf{s}_{t+h-1}|\mathbf{S}_t\right)f_{\mathbf{s}}\left(E\left(\mathbf{Y}_{t+h-1}|\mathbf{S}_t\right),\mathbf{A}^{h-1}\mathbf{s}_t,\mathbf{0}\right)\\
&+f_{\boldsymbol{\varepsilon}}^{'}\left(E\left(\mathbf{Y}_{t+h-1}|\mathbf{S}_t\right),\mathbf{A}^{h-1}\mathbf{s}_t,\mathbf{0}\right)\Sigma_{\varepsilon}f_{\boldsymbol{\varepsilon}}\left(E\left(\mathbf{Y}_{t+h-1}|\mathbf{S}_t\right),\mathbf{A}^{h-1}\mathbf{s}_t,\mathbf{0}\right).
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
Var\left(\mathbf{s}_{t+h-1}|\mathbf{S}_t\right) &= \mathbf{0}_{k\times k} \text{ if } h=1\\
Var\left(\mathbf{s}_{t+h-1}|\mathbf{S}_t\right) &= \sum_{i=1}^{h-1}\left(\mathbf{A}^{i-1}\right)\Sigma_{\varepsilon}\left(\mathbf{A}^{i-1}\right)^{'} \text{ for } h>1
\end{aligned} \tag{9}$$

Proof: Iterating the motion equation (2) forward we have:

$$\mathbf{s}_{t+h-1} = \mathbf{A}^{h-1}\mathbf{s}_t + \sum_{i=1}^{h-1} \mathbf{A}^{i-1}\boldsymbol{\varepsilon}_{t+h-i} \tag{10}$$

So,

$$E\left(\mathbf{s}_{t+h-1}|\mathbf{S}_t\right) = \mathbf{A}^{h-1}\mathbf{s}_t. \tag{11}$$

For $h = 1$:

$$Var\left(\mathbf{s}_{t+h-1}|\mathbf{S}_t\right) = Var\left(\mathbf{s}_t|\mathbf{S}_t\right) = \mathbf{0}_{k\times k} \tag{12}$$

For $h > 1$:

$$Var\left(\mathbf{s}_{t+h-1}|\mathbf{S}_t\right) = \sum_{i=1}^{h-1} \left(\mathbf{A}^{i-1}\right)\Sigma_\varepsilon\left(\mathbf{A}^{i-1}\right)'. \tag{13}$$

In addition, $\boldsymbol{\varepsilon}$ is the vector of innovations, so $E\left(\boldsymbol{\varepsilon}_{t+h}|\mathbf{S}_t\right) = \mathbf{0}_{k\times 1}$ and $Var\left(\boldsymbol{\varepsilon}_{t+h}|\mathbf{S}_t\right) = \Sigma_\varepsilon$.
■

To compute the variance contributed by shock j at horizon h, called $Var^j\left(Y_{t+h}|\mathbf{S}_t\right)$, we set the variances of all the other shocks $i \neq j$ to zero. In particular, let Σ_ε^j be a variance-covariance matrix where all elements are zero, except element $(j,j) = Var\left(\varepsilon_t^j\right)$. We can compute $Var^j\left(Y_{t+h}|\mathbf{S}_t\right)$ based on equation (8), except using Σ_ε^j instead of Σ_ε.

The variance decomposition for shock j at horizon h conditional on state $\mathbf{S}_t$ can be computed as follows:

$$\lambda^j\left(h|\mathbf{S}_t\right) = \frac{Var^j\left(Y_{t+h}|\mathbf{S}_t\right)}{\sum_{i=1}^{k} Var^i\left(Y_{t+h}|\mathbf{S}_t\right)}. \tag{14}$$

2.3 Lanne and Nyberg (2006) method

In this subsection, we summarize the method proposed by Lanne and Nyberg (2016), so the reader can compare their method and our methods. In the Application section below, we compare the results of these three methods using an empirical model.

According to Lanne and Nyberg (2016), forecast error variance decomposition can be computed based on generalized impulse response function (GIRF), which is proposed by Koop et al. (1996). In particular, define the GIRF for variable Y at horizon h under a shock of magnitude δ_j to ε_{jt}, conditional on state $\mathbf{S}_t$:

$$GI^j\left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j\right) = E\left[Y_{t+h}|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j\right] - E\left[Y_{t+h}|\mathbf{S}_t\right] \tag{15}$$

where the first expectation is taken over the distribution of $\left\{\boldsymbol{\varepsilon}_{t+1}^{i\neq j}, \boldsymbol{\varepsilon}_{t+2}, ..., \boldsymbol{\varepsilon}_{t+h}\right\}$ and the second expectation is taken with respect to $\{\boldsymbol{\varepsilon}_{t+l}\}_{l=1}^{h}$.

They propose that we compute the forecast error variance decomposition of shock j conditional on state $\mathbf{S}_t$:

$$\lambda^j\left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j\right) = \frac{\sum_{l=1}^{h}\left(GI^j\left(l|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j\right)\right)^2}{\sum_{i=1}^{m}\sum_{l=1}^{h}\left(GI^i\left(l|\mathbf{S}_t, \varepsilon_{it+1} = \delta_i\right)\right)^2}. \tag{16}$$

This formula is very similar the traditional forecast error variance decomposition for a linear VAR model. The only small modification is that they use generalized impulse responses in replace of traditional impulse responses.

Proof: The traditional impulse responses are computed as:

$$\begin{aligned} I^j\left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j\right) &= E\left[Y_{t+h}|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j, \varepsilon_{t+1}^{i\neq j} = 0, \{\boldsymbol{\varepsilon}_{t+l}\}_{l=2}^{h}\right] \\ &\quad - E\left[Y_{t+h}|\mathbf{S}_t, \{\boldsymbol{\varepsilon}_{t+l}\}_{l=1}^{h} = 0\right] \end{aligned} \tag{17}$$

The the h-period dynamic multiplier is defined as:

$$m^j\left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j\right) = \frac{I^j\left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j\right)}{\delta_j}, \tag{18}$$

and the FEVD is computed as:

$$\lambda^j\left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j\right) = \frac{\sum_{l=1}^{h}\left(\sigma_j^2 m^j\left(l|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j\right)\right)^2}{\sum_{i=1}^{m}\sum_{l=1}^{h}\left(\sigma_i^2 m^i\left(l|\mathbf{S}_t, \varepsilon_{it+1} = \delta_i\right)\right)^2} \tag{19}$$

For a linear VAR model with orthogonal shocks, $m^j\left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j\right)$ does not depend on state $\mathbf{S}_t$ and shock δ_j. As a result, $m^j\left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j\right) = m^j\left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \sigma_j\right)$, where σ_j is the standard deviation of shock j. Applying this result in equation 19, we obtain:

$$\lambda^j\left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j\right) = \frac{\sum_{l=1}^{h}\left(I^j\left(l|\mathbf{S}_t, \varepsilon_{jt+1} = \sigma_j\right)\right)^2}{\sum_{i=1}^{m}\sum_{l=1}^{h}\left(I^i\left(l|\mathbf{S}_t, \varepsilon_{it+1} = \sigma_i\right)\right)^2}. \tag{20}$$

■

Therefore, we should compute the variance decomposition attributed to shock j at horizon h conditional on state $\mathbf{S}_t$:

$$\lambda^j (h|\mathbf{S}_t) = E\left(\lambda^j (h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j)\right), \tag{21}$$

where the expectation is taken over the distribution of δ_j.

Among these three methods, the total variance method is the only method that provides **true** FEVD. Both the Delta method and the Lanne and Nyberg (2016) provide an approximation of the true FEVD. While our two methods are applications of the law of total variance and the Delta method, the Lanne and Nyberg (2016) method does not have a clear theoretical background.

3 Application

In this section we apply our methods and the Lanne and Nyberg method to compute and analyze the FEVD for a conventional fully-nonlinear DSGE model with occasionally binding ZLB. We also provide our algorithms to implement these methods. In addition, we provide a comparison between the Lanne and Nyberg method and our methods.

3.1 Model

The model is the conventional dynamic New-Keynesian (DNK) model with Rotemberg price adjustments. It consists of a continuum of identical households, a continuum of identical competitive final good producers, a continuum of monopolistically competitive intermediate goods producers, and a government (monetary and fiscal authorities).

3.1.1 Households

The representative household maximizes his expected discounted utility

$$E_1 \left\{ \sum_{t=1}^{\infty} \left(\Pi_{j=0}^{t-1} \beta_j\right) \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \chi \frac{N_t^{1+\eta}}{1+\eta} \right) \right\} \tag{22}$$

subject to the budget constraint

$$P_t C_t + (1+i_t)^{-1} B_t = W_t N_t + B_{t-1} + \Pi_t + T_t, \tag{23}$$

where C_t is consumption of final goods, i_t is the nominal interest rate, B_t denotes one-period bond holdings, N_t is labor, W_t is the nominal wage rate, Π_t is the profit income, T_t is the lump-sum tax, and β_t denotes the preference shock. I assume that β_t follows an AR(1) process

$$\ln(\beta_t) = (1-\rho_\beta)\ln\beta + \rho_\beta \ln\left(\beta_{t-1}\right) + \varepsilon_{\beta t},\ \ \beta_0 = 1 \tag{24}$$

where $\rho_\beta \in (0,1)$ is the persistence of the preference shock and $\varepsilon_{\beta t}$ is the innovation of the preference shock with mean 0 and variance σ_β^2. The preference shock is a reduced form of more realistic forces that can drive the nominal interest rate to the ZLB.[4]

The first-order conditions for the household optimization problem are given by

$$\chi N_t^\eta C_t^\gamma = w_t, \tag{25}$$

and

$$E_t\left[M_{t,t+1}\left(\frac{1+i_t}{1+\pi_{t+1}}\right)\right] = 1, \tag{26}$$

where $w_t = W_t/P_t$ is the real wage, $\pi_t = P_t/P_{t-1} - 1$ is the inflation rate, and the stochastic discount factor is given by

$$M_{t,t+1} = \beta_t \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}. \tag{27}$$

3.1.2 Final good producers

To produce the final good, the final good producers buy and aggregate a variety of intermediate goods indexed by $i \in [0,1]$ using a CES technology

$$Y_t = \left(\int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}},$$

where ε is the elasticity of substitution among intermediate goods. The profit maximization problem is given by

$$\max\ P_t Y_t - \int_0^1 P_t(i)\, Y_t(i)\, di,$$

[4]This setting is very common in the ZLB literature, for example see Nakata (2011) and Ngo (2014b) among others. Another way to make the ZLB binding is to introduce a deleveraging shock as in Eggertsson and Krugman (2012), Guerrieri and Lorenzoni (2011), and Ngo (2015).

where $P_t(i)$ and $Y_t(i)$ are the price and quantity of intermediate good i. Profit maximization and the zero-profit condition give the demand for intermediate good i

$$Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} Y_t, \tag{28}$$

and the aggregate price level

$$P_t = \left(\int P_t(i)^{1-\epsilon}\, di\right)^{\frac{1}{1-\epsilon}}. \tag{29}$$

3.1.3 Intermediate goods producers

There is a unit mass of intermediate goods producers on $[0,1]$ that are monopolistic competitors. Suppose that each intermediate good $i \in [0,1]$ is produced by one producer using the linear technology

$$Y_t(i) = A_t N_t(i), \tag{30}$$

where $N_t(i)$ is labor input and A_t denotes the supply shock that follows an AR(1) process:

$$\ln A_t = \rho_A \ln A_{t-1} + \varepsilon_{At}, \tag{31}$$

where $\rho_A \in (0,1)$ is the persistence parameter and ε_{At} is the innovation with mean 0 and variance σ_A^2. Cost minimization implies that each firm faces the same real marginal cost divided by productivity:

$$mc_t = mc_t(i) = \frac{w_t}{A_t}. \tag{32}$$

3.1.4 Price setting

Following Rotemberg (1982), we assume that each intermediate goods firm i faces costs of adjusting prices in terms of final goods. In this paper, we use a quadratic adjustment cost function, which was proposed by Ireland (1997) and which is one of the most common functions used in the ZLB literature:

$$\frac{\varphi}{2}\left(\frac{P_t(i)}{P_{t-1}(i)} - 1\right)^2 Y_t,$$

where φ is the adjustment cost parameter which determines the degree of nominal price rigidity.[5] The problem of firm i is given by

$$\max_{\{P_t(i)\}} E_t \sum_{j=0}^{\infty} \left\{ M_{t,t+j} \left[\left(\frac{P_{t+j}(i)}{P_{t+j}} - mc_t \right) Y_{t+j}(i) - \frac{\varphi}{2} \left(\frac{P_{t+j}(i)}{P_{t+j-1}(i)} - 1 \right)^2 Y_{t+j} \right] \right\} \tag{33}$$

subject to its demand (28). In a symmetric equilibrium, all firms will choose the same price and produce the same quantity, i.e., $P_t(i) = P_t$ and $Y_t(i) = Y_t$. The optimal pricing rule then implies that

$$\left(1 - \varepsilon + \varepsilon \frac{w_t}{A_t} - \varphi \pi_t (1 + \pi_t) \right) Y_t + \varphi E_t \left[M_{t,t+1} \pi_{t+1} (1 + \pi_{t+1}) Y_{t+1} \right] = 0. \tag{34}$$

3.1.5 Monetary and fiscal policies

The central bank conducts monetary policy by setting the interest rate using a simple Taylor rule, subject to the ZLB condition:

$$\frac{1 + i_t}{1 + i} = \max \left\{ \left(\frac{GDP_t}{GDP} \right)^{\phi_y} \left(\frac{1 + \pi_t}{1 + \pi} \right)^{\phi_\pi}, \frac{1}{1 + i} \right\} \tag{35}$$

where $GDP_t \equiv C_t + G_t$ denotes the gross domestic product (GDP); $G_t = 0.2GDP_t$; and GDP, π, and i denote the steady state GDP level, the targeted inflation rate, and the steady-state nominal interest rate, respectively.[6]

3.2 Equilibrium systems

With the Rotemberg price setting, the aggregate output satisfies

$$Y_t = A_t N_t, \tag{36}$$

and the resource constraint is given by

$$C_t + G_t + \frac{\varphi}{2} \pi_t^2 Y_t = Y_t. \tag{37}$$

The equilibrium system for the Rotemberg model consists of a system of six nonlinear difference equations (25), (26), (34), (35), (36), (37) for six variables w_t, C_t, i_t, π_t, N_t, and Y_t.

[5] For example, see Nakata (2011) and Aruoba and Schorfheide (2013) among others. It would also be interesting to compare the time-dependent Calvo price setting to another state-dependent price setting as in Dotsey et al. (1999) and Ngo (2014a) at the ZLB.

[6] Some researchers use the flexible price equilibrium output as the output target in the Taylor rule, and some researchers also include the lagged interest rate. These alternative specifications will not change our key insights.

Table 1: CALIBRATION

Symbol	Description	Values
β	Quarterly discount factor	0.995
γ	CRRA parameter	1
η	Inverse labor supply elasticity	1
ε	Monopoly power	7.66
φ	Price adjustment cost parameter in the Rotemberg model	495.8
π	Inflation target	0
ϕ_π	Weight of inflation target in the Taylor rule	1.75
ϕ_y	Weight of output target in the Taylor rule	$\frac{0.5}{4}$
σ_β	Standard deviation of preference innovations	$\frac{0.13}{100}$
ρ_β	AR-coefficient of preference shocks	0.85
σ_A	Standard deviation of technology innovations	$\frac{0.25}{100}$
ρ_A	AR-coefficient of government spending shocks	0.90

3.3 Calibration

We calibrate the primitive parameters of the model based on the existing literature.[7] The quarterly subjective discount factor β is set at 0.995 such that the annual real interest rate is 2%, as in Christiano et al. (2011) and Boneva et al. (2016). The constant relative risk aversion parameter γ is 1, corresponding to a log utility function with respect to consumption. This utility function is commonly used in the business cycles literature. The labor supply elasticity with respect to wages is set at 1, or $\eta = 1$, as in Christiano et al. (2011). The value of χ is calibrated to obtain the steady state faction of working hours of 1/3. The elasticity of substitution among differentiated intermediate goods ϵ is 7.66, corresponding to a 15% net markup. This value is also popular in the literature (e.g. Boneva et al. (2016)).

We set the price adjustment cost parameter in the Rotemberg model $\varphi = 495.8$ as in Boneva et al. (2016). This value, together with the other parameters, implies that the slope of the Phillips curve is 0.0269, which is within the range estimated by Ball and Mazumder (2011) for the U.S. using the 1985:q1-2007:q4 data.

We set the parameters in the Taylor rule at $\phi_\pi = 1.75$ and $\phi_y = 0.25$, which are close

[7]Another way to assign values to the primitive parameters of the model is to estimate the model and let the data speak out. However, estimating a fully nonlinear DSGE model with occasionally-binding ZLB is very computationally expensive, and is not a contribution of this paper. Therefore, instead of estimating the model, we calibrate the parameters based on the existing literature.

to the estimates by Gust et al. (2017).

Following Fernandez-Villaverde et al. (2015), we set the persistence of technology shock $\rho_A = 0.9$ and the standard deviation for the shock innovations $\sigma_A = \frac{0.25}{100}$. They argue that this technology shock specification explains the U.S. data reasonably well for the past two decades.

The most important parameters left to calibrate are those regarding the preference shock specification. Following Gust et al. (2017), we set the persistence of preference shock at 0.85. We set the standard deviation for preference innovations $\sigma_\beta = \frac{0.13}{100}$ so that the unconditional probability of hitting the ZLB is 17%, which is consistent with the recent U.S. data. In particular, using the method used in Ball (2013) and Ngo (2016), we find that the probability of nominal interest rate hitting the ZLB would be between $16.1 - 19.7\%$ if the Fed kept the inflation target as low as 2%, see Appendix for more detail. In addition, in the appendix we show that the second moments of simulated data fit the U.S. data very well.

3.4 Solution and algorithms

3.4.1 Solution method

In terms of numerical solution, we use projection methods, which is similar to Ngo (2014b). In particular, we solve for policy function using a finite element method called the linear spline interpolation (see Judd (1998) and Miranda and Fackler (2002)). The policy can be cast in the form of equation (1), as described in Section 2. In particular, let $\mathbf{Y} = (i, C, N, Y, GDP, w, \pi)'$, where i, C, N, Y, GDP, w, and π denote the nominal interest rate, consumption, labor, output, GDP, the wage rate, and the inflation rate, respectively. Then,

$$\mathbf{Y}_t = \mathbf{f}\left(\mathbf{S}_{t-1}, \boldsymbol{\varepsilon}_t\right), \tag{38}$$

where $\mathbf{f}(\cdot) : R^{11} \longrightarrow R^7$ is a known nonlinear function that we have solved; $\mathbf{S}_{t-1} = (\mathbf{Y}_{t-1}; \mathbf{s}_{t-1})$ is the vector of endogenous and exogenous state variables; the vector of exogenous state variables $\mathbf{s}_t = (A_{t-1}, \beta_{t-1})'$ has the following transition equation:

$$\mathbf{s}_t = \mathbf{A}\mathbf{s}_{t-1} + \boldsymbol{\varepsilon}_t; \tag{39}$$

$\mathbf{A} = \begin{pmatrix} \rho_A & 0 \\ 0 & \rho_\beta \end{pmatrix}$; $\boldsymbol{\varepsilon}_t = (\varepsilon_{At}, \varepsilon_{\beta t})'$ is the 2×1 vector of orthorgonal supply (technology) and demand (preference) shocks with a known diagonal variance-covariance matrix $\Sigma_\varepsilon =$

$\begin{pmatrix} \sigma_A^2 & 0 \\ 0 & \sigma_\beta^2 \end{pmatrix}$ and mean $\mathbf{0}_{2\times 1}$.[8]

In addition to the policy function, we also solve for its partial derivatives with respect to state. Base on the policy function and its partial derivatives, we run simulation and compute variance decomposition for variables of interest using the three methods described in Section 2. In the rest of this subsection, we provide three algorithms for these three methods. To be consistent with the notation in Section 2, let us use Y to denote any single component of $\mathbf{Y}$.

3.4.2 Algorithm for the total variance method

To compute FEVD for Y based on the total variance method, we use the following algorithm:

Algorithm 1 *Total variance method*

- *Step 1: Simulate N_2 paths of $\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}$.*
- *Step 2: For each path of $\left\{\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right\}^{q}$ for $q = 1,..,N_2,$ simulate N_1 paths of $\left\{\varepsilon_{t+l}^{j}\right\}_{l=1}^{h}$. Then compute the expectation and variance of Y_{t+h} contributed by shock j conditional on $\mathbf{S}_t$ and $\left\{\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right\}^{q}$:*

$$\widehat{E}\left(Y_{t+h}|\mathbf{S}_t,\left\{\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right\}^{q}\right) = \frac{1}{N_1}\sum_{m=1}^{N_1} Y_{t+h}|\mathbf{S}_t,\left\{\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right\}^{q},\left\{\left\{\varepsilon_{t+l}^{j}\right\}_{l=1}^{h}\right\}^{m} \tag{40}$$

$$\widehat{Var}\left(Y_{t+h}|\mathbf{S}_t,\left\{\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right\}^{q}\right) = \frac{1}{N_1}\sum_{m=1}^{N_1}\left(\begin{array}{c} Y_{t+h}|\mathbf{S}_t,\left\{\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right\}^{q},\left\{\left\{\varepsilon_{t+l}^{j}\right\}_{l=1}^{h}\right\}^{m} \\ -\widehat{E}\left(Y_{t+h}|\mathbf{S}_t,\left\{\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right\}^{q}\right)\end{array}\right)^{2} \tag{41}$$

[8] In this standard DNK model, there is not any endogenous state variable $\mathbf{Y}_{t-1}$. Therefore, the coefficients associated with these edogenous variables in the policy function are zero. However, we keep using $\mathbf{Y}_{t-1}$ to ensure the generality of the equations and the algorithms below.

- *Step 3: Compute the variance of Y_{t+h} conditional on state $\mathbf{S}_t$:*

$$\widehat{Var}\left(Y_{t+h}|\mathbf{S}_t\right) = \widehat{E}\left(Var\left(Y_{t+h}|\mathbf{S}_t,\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right)\right) + \widehat{Var}\left(E\left(Y_{t+h}|\mathbf{S}_t,\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right)\right) \quad (42)$$

where the variance caused by shock j is:

$$\widehat{Var}^{j}\left(Y_{t+h}|\mathbf{S}_t\right) = \widehat{E}\left(Var\left(Y_{t+h}|\mathbf{S}_t,\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right)\right) \quad (43)$$

$$= \frac{1}{N_2}\sum_{q=1}^{N_2}\widehat{Var}\left(Y_{t+h}|\mathbf{S}_t,\left\{\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right\}^{q}\right) \quad (44)$$

and the variance caused by the other shocks:

$$\widehat{Var}^{i\neq j}\left(Y_{t+h}|\mathbf{S}_t\right) = Var\left(E\left(Y_{t+h}|\mathbf{S}_t,\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right)\right) = \frac{1}{N_2}\sum_{q=1}^{N_2}\left(\begin{array}{c}\widehat{E}\left(Y_{t+h}|\mathbf{S}_t,\left\{\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right\}^{q}\right) \\ -\overline{\widehat{E}\left(Y_{t+h}|\mathbf{S}_t,\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right)}\end{array}\right)^2$$

where

$$\overline{\widehat{E}\left(Y_{t+h}|\mathbf{S}_t,\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right)} = \frac{1}{N_2}\sum_{q=1}^{N_2}\widehat{E}\left(Y_{t+h}|\mathbf{S}_t,\left\{\left\{\boldsymbol{\varepsilon}_{t+l}^{i\neq j}\right\}_{l=1}^{h}\right\}^{q}\right) \quad (45)$$

- *Step 3: Compute forecast error variance decomposition (FEVD) for shock j at horizon h conditional on state $\mathbf{S}_t$:*

$$\widehat{\lambda}^{j}\left(h|\mathbf{S}_t\right) = \frac{\widehat{Var}^{j}\left(Y_{t+h}|\mathbf{S}_t\right)}{\widehat{Var}\left(Y_{t+h}|\mathbf{S}_t\right)}. \quad (46)$$

Then compute the modified FEVD:

$$\widetilde{\widehat{\lambda}}^{j}\left(h|\mathbf{S}_t\right) = \frac{\widehat{\lambda}^{j}\left(h|\mathbf{S}_t\right)}{\sum_{i=1}^{m}\widehat{\lambda}^{i}\left(h|\mathbf{S}_t\right)}. \quad (47)$$

3.4.3 Algorithm for the Delta method

To compute the variance decomposition using the Delta method, we use the following algorithm:

Algorithm 2 *The Delta method*

- *Step 1: Compute the variance of Y_{t+h} contributed by shock j conditional on state $\mathbf{S}_t$.*

  Let Σ_ε^j be a variance-covariance matrix where all elements are zero, except element $(j,j) = Var(\varepsilon_{jt})$. Compute the variance contributed by shock j at horizon h using the following recursion:

  For $h = 1$ compute:

$$\widehat{Var}^j (Y_{t+1}|\mathbf{S}_t) = f'_{\boldsymbol{\varepsilon}} (\mathbf{Y}_t, \mathbf{s}_t, \mathbf{0}) \Sigma_\varepsilon^j f_{\boldsymbol{\varepsilon}} (\mathbf{Y}_t, \mathbf{s}_t, \mathbf{0}) .$$

  For any $h > 1$, compute:

$$\widehat{Var}^j (\mathbf{s}_{t+h-1}|\mathbf{S}_t) = \sum_{i=1}^{h-1} \left(\mathbf{A}^{i-1}\right) \Sigma_\varepsilon^j \left(\mathbf{A}^{i-1}\right) \tag{48}$$

  and

$$\begin{aligned} \widehat{Var}^j (Y_{t+h}|\mathbf{S}_t) \quad = \quad & f'_{\mathbf{Y}} \left(\widehat{E} (\mathbf{Y}_{t+h-1}|\mathbf{S}_t), \mathbf{A}^{h-1}\mathbf{s}_t, \mathbf{0}\right) \widehat{Var}^j (\mathbf{Y}_{t+h-1}|\mathbf{S}_t) f_{\mathbf{Y}} \left(\widehat{E} (\mathbf{Y}_{t+h-1}|\mathbf{S}_t), \mathbf{A}^{h-1}\mathbf{s}_t, \mathbf{0}\right) \\ & + f'_{\mathbf{s}} \left(\widehat{E} (\mathbf{Y}_{t+h-1}|\mathbf{S}_t), \mathbf{A}^{h-1}\mathbf{s}_t, \mathbf{0}\right) \widehat{Var}^j (\mathbf{s}_{t+h-1}|\mathbf{S}_t) f_{\mathbf{s}} \left(\widehat{E} (\mathbf{Y}_{t+h-1}|\mathbf{S}_t), \mathbf{A}^{h-1}\mathbf{s}_t, \mathbf{0}\right) \\ & + f'_{\boldsymbol{\varepsilon}} \left(\widehat{E} (\mathbf{Y}_{t+h-1}|\mathbf{S}_t), \mathbf{A}^{h-1}\mathbf{s}_t, \mathbf{0}\right) \Sigma_\varepsilon^j f_{\boldsymbol{\varepsilon}} \left(\widehat{E} (\mathbf{Y}_{t+h-1}|\mathbf{S}_t), \mathbf{A}^{h-1}\mathbf{s}_t, \mathbf{0}\right), \end{aligned} \tag{49}$$

  where

$$\widehat{E} (\mathbf{Y}_{t+h-1}|\mathbf{S}_t) = \frac{1}{N_3} \sum_{m=1}^{N_3} \mathbf{Y}_{t+h}|\mathbf{S}_t, \left\{\{\boldsymbol{\varepsilon}_{t+l}\}_{l=1}^{h}\right\}^m . \tag{50}$$

- *Step 2: Compute the variance decomposition for shock j at horizon h conditional on state $\mathbf{S}_t$:*

$$\widehat{\lambda}^j (h|\mathbf{S}_t) = \frac{\widehat{Var}^j (Y_{t+h}|\mathbf{S}_t)}{\sum_{i=1}^{k} \widehat{Var}^i (Y_{t+h}|\mathbf{S}_t)} \tag{51}$$

3.4.4 Algorithm for the Lanne and Nyberg method

To compute the variance decomposition using the Lanne and Nyberg (2016) method, we use the following algorithm:

Algorithm 3 *The Lanne and Nyberg method*

- *Step 1: Draw a set of* N_5 *vectors of shocks:* $(\delta_1^m, .., \delta_k^m)$ *for* $m = 1, .., N_5$.
- *Step 2: For each* $(\delta_1^m, .., \delta_k^m)$, *simulate* N_4 *paths of* $\{\boldsymbol{\varepsilon}_{t+l}\}_{l=1}^h$ *. Then, compute generalized impulse responses for variable* Y_{t+h} *under a shock to* ε_{jt+1}, *conditional on state* $\mathbf{S}_t$ *and shock size* δ_j^m*:*

$$\widehat{GI}^j \left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j^m\right) = \widehat{E}\left[Y_{t+h}|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j^m\right] - \widehat{E}\left[Y_{t+h}|\mathbf{S}_t\right], \tag{52}$$

where

$$\begin{aligned} \widehat{E}\left[Y_{t+h}|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j^m\right] &= \frac{1}{N_4}\sum_{q=1}^{N_4} Y_{t+h}|\mathbf{S}_t, \varepsilon_{jt+1} = \sigma_j, \left\{\varepsilon_{t+1}^{i\neq j}\right\}^q, \left\{\{\boldsymbol{\varepsilon}_{t+l}\}_{l=2}^h\right\}^q, \\ \widehat{E}\left[Y_{t+h}|\mathbf{S}_t\right] &= \frac{1}{N_4}\sum_{q=1}^{N_4} Y_{t+h}|\mathbf{S}_t, \left\{\{\boldsymbol{\varepsilon}_{t+l}\}_{l=1}^h\right\}^q. \end{aligned}$$

- *Step 2: Compute the FEVD for shock* j *to* Y_{t+h} *conditional on state* $\mathbf{S}_t$ *and shock size* δ_j^m*:*

$$\widehat{\lambda}^j \left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j^m\right) = \frac{\sum_{l=1}^{h}\left(\widehat{GI}^j \left(l|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j^m\right)\right)^2}{\sum_{i=1}^{k}\sum_{l=1}^{h}\left(\widehat{GI}^i \left(l|\mathbf{S}_t, \varepsilon_{it+1} = \delta_i^m\right)\right)^2}. \tag{53}$$

- *Step 3: Compute the FEVD for shock* j *to* Y_{t+h} *conditional on state* $\mathbf{S}_t$ *:*

$$\widehat{\lambda}^j \left(h|\mathbf{S}_t\right) = \frac{1}{N_5}\sum_{m=1}^{N_5} \widehat{\lambda}^j \left(h|\mathbf{S}_t, \varepsilon_{jt+1} = \delta_j^m\right).$$

3.5 Variance decomposition results

In this section we compute and report FEVD for two cases: at steady state and at the state that mimics the Great Recession. At the state that mimics the Great Recession, GDP declines by about 6.5% quarterly, the deflation rate is about 1.2% annually, and the median of ZLB duration is about 5 quarters.[9] Although the GDP growth rate and the ZLB duration are in line with the US data, the deflation rate is much smaller than the actual U.S. data. In particular, the U.S. GDP declined by 6.5% and the deflation rate was 3.5% per quarter (or 14.0% annually) at the trough of the Great Recession.[10]

3.5.1 Variance decomposition at steady state

Table 2 shows the variance decomposition for GDP and inflation at different horizons starting from steady state, based on the total variance method, the Delta method, and the Lanne and Nyberg (2016) method. For the total variance method we use $N_1 = N_2 = 1,999$. For the Delta method, we use $N_3 = 9,999$. For the Lanne and Nyberg method, we use $N_4 = N_5 = 4,999$. Increasing these simulation sizes further will not change our main results. Again, among these methods, only the total variance method provides the **true** FEVD when the simulation sample is sufficiently large. Both the Delta method and the Lanne and Nyberg (2016) method are an approximation of the total variance method.

Given the parameter calibration, the total variance method shows that at steady state the technology shock contributes 11.8% and 25.8% to the forecast error variance of GDP and inflation at 1-period horizon, respectively. The contribution declines at longer horizon. In particular, at 20-period horizon the contribution of technology shock to GDP and inflation increases to 13.2% and 34.8%, respectively. Based on the parameterization, the preference shock dominates the technology shock in explaining the volatility of GDP and inflation at all horizons.

It is very important to note that compared to the Lanne ad Nyberg method, the Delta method produces the results closer to those from the total variance method, which is considered as the only method providing the **true** results, especially at longer horizons. For example, at 12-period horizon both the total variance and the Delta method estimate the contribution of technology shocks to GDP of around $13.0 - 15.5\%$, while the Lanne

[9] Based on our simulation, there are many simulated series where the ZLB binds more than 30 periods consecutively.

[10] See Appendix for more detailed information regarding the GDP growth rate and the deflation rate in the U.S. at the trough of the Great Recession.

Table 2: Variance decomposition at steady state

Panel A. GDP

	Total variance		Delta method		Lanne and Nyberg (2016)	
h	Technology	Preference	Technology	Preference	Technology	Preference
1	0.1182	0.8818	0.1840	0.8160	0.0019	0.9981
2	0.1170	0.8830	0.1840	0.8160	0.0018	0.9982
3	0.1190	0.8810	0.1794	0.8206	0.0016	0.9984
4	0.1125	0.8875	0.1749	0.8251	0.0014	0.9986
8	0.1288	0.8712	0.1630	0.8370	0.0013	0.9987
12	0.1126	0.8874	0.1583	0.8417	0.0014	0.9986
16	0.1238	0.8762	0.1568	0.8432	0.0014	0.9986
20	0.1321	0.8679	0.1563	0.8437	0.0014	0.9986

Panel B. Inflation

	Total variance		Delta method		Lanne and Nyberg (2016)	
h	Technology	Preference	Technology	Preference	Technology	Preference
1	0.2574	0.7426	0.3489	0.6511	0.1401	0.8599
2	0.2709	0.7291	0.3489	0.6511	0.1360	0.8640
3	0.2869	0.7131	0.3420	0.6580	0.1329	0.8671
4	0.2850	0.7150	0.3350	0.6650	0.1307	0.8693
8	0.3306	0.6694	0.3164	0.6836	0.1271	0.8729
12	0.3129	0.6871	0.3090	0.6910	0.1263	0.8737
16	0.3363	0.6637	0.3065	0.6935	0.1262	0.8738
20	0.3475	0.6525	0.3058	0.6942	0.1261	0.8739

and Nyberg (2016) method produces a contribution of only 0.14%. For the other horizons, the Delta method also produce better results for both GDP and inflation than the Lanne and Nyberg method does.

3.5.2 Variance decomposition at the Great Recession

Table 3 shows the variance decomposition for GDP and inflation at different horizons starting from a state that mimics the Great Recession, based on the total variance method, the Delta method, and the Lanne and Nyberg (2016) method.

Table 3: Variance decomposition at the Great Recession

Panel A. GDP

	Total variance		Delta method		Lanne and Nyberg (2016)	
h	Technology	Preference	Technology	Preference	Technology	Preference
1	0.0091	0.9909	0.0121	0.9879	0.0590	0.9410
2	0.0087	0.9913	0.0089	0.9911	0.0561	0.9439
3	0.0093	0.9907	0.0049	0.9951	0.0509	0.9491
4	0.0106	0.9894	0.0019	0.9981	0.0474	0.9526
8	0.0251	0.9749	0.0592	0.9408	0.0432	0.9568
12	0.0383	0.9617	0.1157	0.8843	0.0429	0.9571
16	0.0553	0.9447	0.1380	0.8620	0.0429	0.9571
20	0.0672	0.9328	0.1470	0.8530	0.0429	0.9571

Panel B. Inflation

	Total variance		Delta method		Lanne and Nyberg (2016)	
h	Technology	Preference	Technology	Preference	Technology	Preference
1	0.2183	0.7817	0.3014	0.6986	0.1454	0.8546
2	0.2173	0.7827	0.3040	0.6960	0.1348	0.8652
3	0.2207	0.7793	0.2995	0.7005	0.1271	0.8729
4	0.2093	0.7907	0.2975	0.7025	0.1228	0.8772
8	0.2224	0.7776	0.2957	0.7043	0.1182	0.8818
12	0.2017	0.7983	0.3009	0.6991	0.1178	0.8822
16	0.2171	0.7829	0.3003	0.6997	0.1177	0.8823

It is interesting to see that the relative importances of technology and preference in explaining the volatility of GDP and inflation change substantially in this Great Recession case, compared to the case of steady state, based on the total variance method and the Delta method. According to the total variance method, the contribution of technology shocks to the volatility of GDP and inflation declines substantially compared to the case of steady state, especially at very short horizons when the nominal interest rate hits the ZLB. For example, based on the total variance method the 1-period-horizon contribution of the technology shock to GDP declines to 0.09% in the Great Recession case, from 11.82% in the steady state case. The 1-period-horizon contribution of the technology shock to inflation also declines significantly to 21.83% from 25.74%.

At longer horizons when the ZLB is no longer binding, the contribution of the technology shock to the volatility of GDP increases, closer to that of the case of steady state. The contribution of technology shock to the variance of GDP at 20-period horizon is around 6.7% based on the total variance method.

To see why the contribution of technology shock to GDP and inflation declines at the Great Recession state, we draw the generalize impulse response function (GIRF) under a shock to technology and preference with magnitudes of one standard deviations (0.57 for technology and 0.25 for preference). Figure 1 shows the GIRFs for two cases: steady state and Great Recession.

Th left column of figure 1 shows the GIRFs for the norminal interest rate, GDP, and the inflation rate at the steady state, while the right column presents the GIRFs at the state that mimics the Great Recession. At the Great Recession state, the median ZLB duration is about 5 quarters. Note that there are many simulated paths where the ZLB binds more than 30 periods consecutively.

It is notable to see that the reponses of economic variables to shocks are magnified at the Great Recession state relative to those at steady state. Especially, the reponse of GDP to positive technology shock becomes smaller, even negative in this case. Note that the response could still be positive if the magnitude of technology shock would be smaller than one standard deviation. Importantly, the reponse of GDP to technology shock becomes much smaller relative to the response of GDP to preference shock at the Great Recession state. This explains why the contribution of technology shock to GDP variance declines substantially at the ZLB compared to at normal times.

In comparing these three methods, from Table 3 we can see that while the Delta method produces the results very similar to those of the total variance method, the Lanne and Nyberg method does not. In particular, if we use the Lanne and Nyberg (2016) method, the relative importance of technology shock does not increase at longer horizons. For example, using the Lanne and Nyberg method, the contributions of the

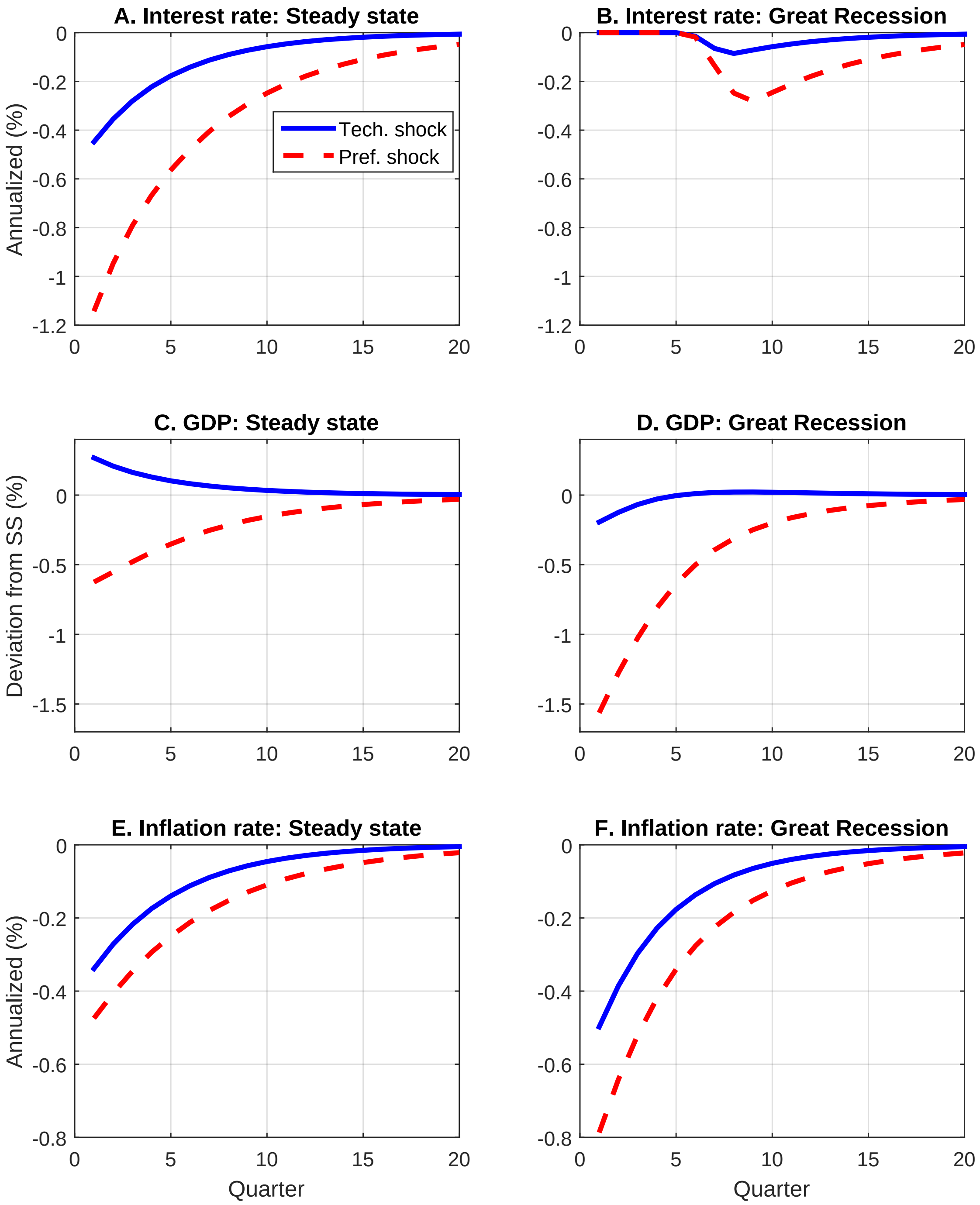

Figure 1: Generalized impulse response function (GIRF). The GIRFs are computed based on one-standard deviation shocks and 4,999 runs, each has 20 periods.

technology shock to the variance of GDP and inflation at 20-period horizon are still relatively small, around 4.29% and 11.77% respectively. These values are smaller than the true results, which come from the total variance method.

In general, the results from the Lanne and Nyberg (2016) method are off the true results generated by the total variance method, and the Delta method is better than the Lanne and Nyberg method. In tables 5 and 6 we provide the results regarding the Lanne and Nyberg method where we compute the GFEVD based on the shock size of one-standard-deviation instead of averaging the GEVD over the distribution of shock sizes. The results based on one-standard-deviation shock size are closer to the true values produced by the total variance method.

4 Conclusion

This paper first proposes two new methods, called the total variance method and the Delta method, to compute and analyze variance decomposition for nonlinear DSGE models. We then apply these methods to a standard DNK model with occasionally zero lower bound (ZLB). Specifically, we study the relative importance of supply and demand shocks to business cycles above and at the ZLB.

We find that the supply shock becomes significantly less important relative to the demand shock in explaining the volatility of economic variables, especially GDP, at a short horizon when the economy stays at a deep recession with binding ZLB. This occurs because the reponse of GDP to technology shock becomes much smaller relative to the response of GDP to preference shock at the Great Recession state with binding ZLB.

In addition, we provide two new algorithms for our new methods. We also compare the results from our new methods to those from the Lanne and Nyberg (2016) method. We find that compared to the Lanne and Nyberg method, the Delta method produces the results closer to the true values coming from the total variance method.

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5 Appendix

5.1 Probability of hitting the ZLB

In this appendix we will answer the question raised in Ball (2013): what would be the unconditional probability of hitting the ZLB have been, had the Fed targeted the inflation rate of 2%. To this end, we follow Ball (2013) and use the real interest rate to answer the question. Specifically, the nominal interest rate equals the real interest rate plus the expected inflation rate. Therefore, we can interpret the zero lower bound on the nominal interest rate as a lower bound of minus expected inflation for the real interest rate. If the target inflation rate is 2%, the expected inflation rate would be 2% and the lower bound on the real interest rate would be -2%. However, Ball (2013) argues that a recession is likely to push expected inflation down somewhat and that the history suggests that the inflation fell about 1% during the past recessions that started with $2-3\%$ inflation rates. Therefore, he finds that the bound on the real interest rate is -1%.

Figure 2 shows: (i) the effective federal funds rate; (ii) the real interest rate computed as the effective federal funds rate minus the inflation rate, where the inflation rate is calculated as a percentage change of the CPI of All Items Less Food and Energy from a year ago; and (iii) the lower bound of the real interest rate. The data spans from 1957:IV, when the data for the CPI of All Items Less Food and Energy was first available, to 2017:II. So, we have 239 observations in all.

From the figure, we are able to see that the real interest rate was smaller than the bound, and, as a result, the nominal interest rate might have hit the ZLB, in the five recessions: 1957:III-1958:II, 1969:IV-1970:IV, 1973:IV-1975:I, 1980:I-1980:IV, and 2007:IV-2009:II.[11] Especially, using the real interest rate, we can very well infer that the nominal interest rate reached the ZLB during the 2007-2009 recession. In addition, the nominal interest rate almost hit the ZLB in the 2001 recession.

Examining the real interest rate since 1957:IV when the CPI data was first available, we find that the ZLB was binding in 47 quarters. Given that the sample has 239 quarters, the unconditional probability of hitting the ZLB is 19.7%. When we computed the real

[11] Ball (2013) argues that in the three out of seven recent recessions excluding the 2007-2009 recession, the nominal interest rate would have hit the ZLB if the inflation rate had been around 2% at the start of the recessions. These three recessions include the 1969-1970 recession, the 1973-1975 recession, and the 1980 recession. Hence, the probability of hitting the ZLB conditional on a recession would be around 50%, or four recessions out of eight recessions, if the Fed targeted 2% inflation rate post World World II.

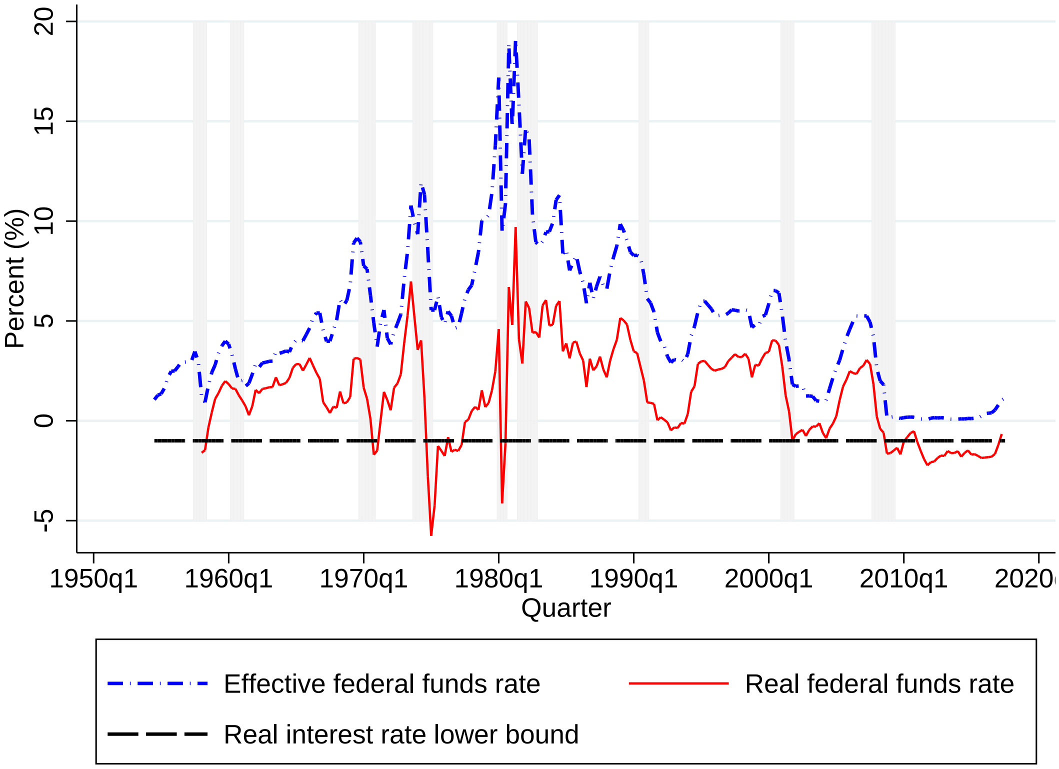

Figure 2: Real federal funds rate is the effective federal funds rate minus the inflation rate computed as a percentage change in the CPI of All Items Less Food and Energy a year ago. The shaded areas indicate the US recessions. Source: the Federal Reserve Economic Data.

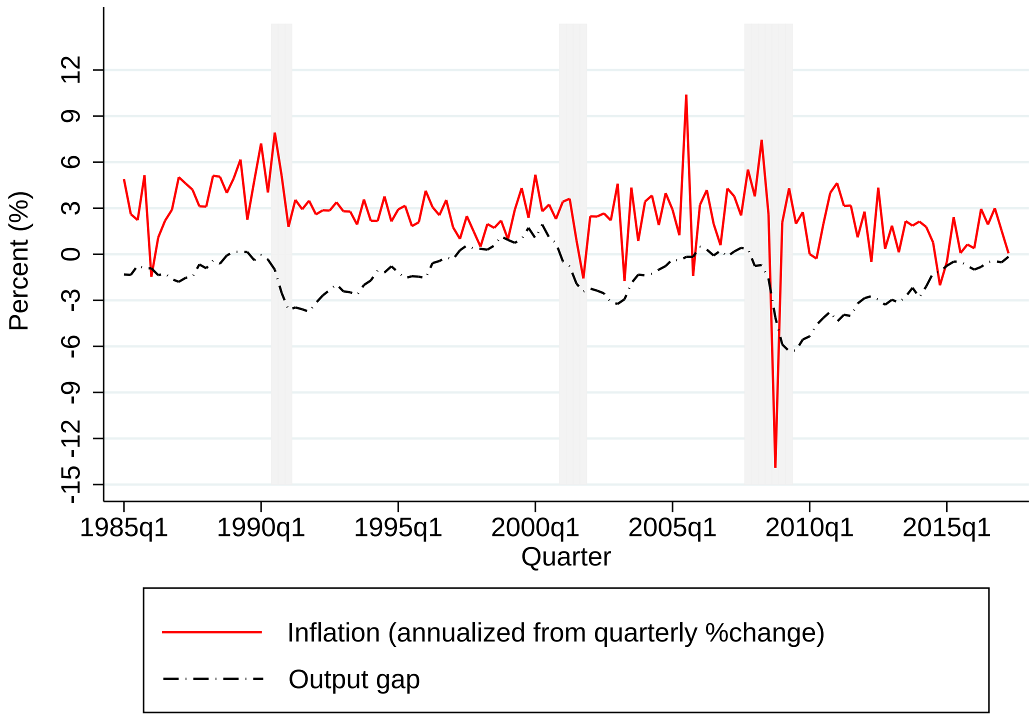

Figure 3: Output gap and inflation in the U.S. Source: the Federal Reserve Economic Data.

interest rate using the CPI of All Items, the probability of hitting the ZLB is slightly smaller, around 16.1%.[12] In this paper, we calibrate the preference shock to match the probability of hitting the ZLB 17%, which is the midpoint of the range $[16.1\%, 19.7\%]$.

5.2 The Great Recession

Figure 3 shows the output gap and inflation series for the U.S. To compute the output gap we take the percentage difference between real GDP and potential real GDP. To compute inflation, we first compute the quarterly percentage change in the CPI, then annualized it by multiplying with 4. The quarterly data on CPI (of All Items), real GDP , and potential real GDP are collected from the Federal Reserve Economic Data website hosted by the Federal Reserve Bank of St. Louis.

[12]For a robust check, we also used the PCE index, instead of the CPI of All Items Less Food and Energy. The result is quite robust.

As seen from this figure, at the trough of the Great Recession, the output gap was as large as around −6.5% (the dash-dotted black line), and the annualized inflation rate (the solid red line) was approximately −14% or −3.5% per quarter. If we use core CPI that excludes food and energy prices, the inflation rate was much smaller, around −0.8%. To be conservative, we target an inflation rate of less than −2% per year in this paper.

5.3 Other tables and figures

Table 4: Comparison of standard deviations

Variable	Data (1957:I-2017:II)	Data (1982:I-2017:II)	Model
GDP growth (%)	0.74	0.65	0.65
Consumption growth (%)	0.80	0.75	0.65
Inflation (%)	2.67	1.37	0.85

The U.S data are from the Federal Reserve Economic Data website, hosted by the Federal Reserve Bank of St. Louis. Inflation is annuallized percentage, computed based on the CPI of All Items Less Food and Energy.

Table 5: Comparison of FVED methods at steady state

Panel A. GDP

	Total variance		Lanne and Nyberg (2016) Averaging over shock sizes		Lanne and Nyberg (2016) Shock size = std. dev. only	
h	Technology	Preference	Technology	Preference	Technology	Preference
1	0.1182	0.8818	0.0019	0.9981	0.1548	0.8452
2	0.1170	0.8830	0.0018	0.9982	0.1416	0.8584
3	0.1190	0.8810	0.0016	0.9984	0.1325	0.8675
4	0.1125	0.8875	0.0014	0.9986	0.1263	0.8737
8	0.1288	0.8712	0.0013	0.9987	0.1139	0.8861
12	0.1126	0.8874	0.0014	0.9986	0.1102	0.8898
16	0.1238	0.8762	0.0014	0.9986	0.1091	0.8909
20	0.1321	0.8679	0.0014	0.9986	0.1087	0.8913

Panel B. Inflation

	Total variance		Lanne and Nyberg (2016) Averaging over shock sizes		Lanne and Nyberg (2016) Shock size = std. dev. only	
h	Technology	Preference	Technology	Preference	Technology	Preference
1	0.2574	0.7426	0.1401	0.8599	0.3374	0.6626
2	0.2709	0.7291	0.1360	0.8640	0.3259	0.6741
3	0.2869	0.7131	0.1329	0.8671	0.3166	0.6834
4	0.2850	0.7150	0.1307	0.8693	0.3090	0.6910
8	0.3306	0.6694	0.1271	0.8729	0.2911	0.7089
12	0.3129	0.6871	0.1263	0.8737	0.2845	0.7155
16	0.3363	0.6637	0.1262	0.8738	0.2824	0.7176
20	0.3475	0.6525	0.1261	0.8739	0.2817	0.7183

Table 6: Comparision of FEVD methods at the Great Recession

Panel A. GDP

	Total variance		Lanne and Nyberg (2016) Averaging over shock sizes		Lanne and Nyberg (2016) Shock size = std. dev. only	
h	Technology	Preference	Technology	Preference	Technology	Preference
1	0.0091	0.9909	0.0590	0.9410	0.0151	0.9849
2	0.0087	0.9913	0.0561	0.9439	0.0128	0.9872
3	0.0093	0.9907	0.0509	0.9491	0.0111	0.9889
4	0.0106	0.9894	0.0474	0.9526	0.0100	0.9900
8	0.0251	0.9749	0.0432	0.9568	0.0088	0.9912
12	0.0383	0.9617	0.0429	0.9571	0.0088	0.9912
16	0.0553	0.9447	0.0429	0.9571	0.0088	0.9912
20	0.0672	0.9328	0.0429	0.9571	0.0088	0.9912

Panel B. Inflation

	Total variance		Lanne and Nyberg (2016) Averaging over shock sizes		Lanne and Nyberg (2016) Shock size = std. dev. only	
h	Technology	Preference	Technology	Preference	Technology	Preference
1	0.2183	0.7817	0.1454	0.8546	0.2860	0.7140
2	0.2173	0.7827	0.1348	0.8652	0.2779	0.7221
3	0.2207	0.7793	0.1271	0.8729	0.2713	0.7287
4	0.2093	0.7907	0.1228	0.8772	0.2664	0.7336
8	0.2224	0.7776	0.1182	0.8818	0.2562	0.7438
12	0.2017	0.7983	0.1178	0.8822	0.2530	0.7470
16	0.2171	0.7829	0.1177	0.8823	0.2520	0.7480
20	0.2196	0.7804	0.1177	0.8823	0.2517	0.7483