Variance Decomposition Analysis for Nonlinear DSGE Models: An Application with ZLB

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Abstract

In this paper, we propose a new method called the total variance method and algorithms to compute and analyze variance decomposition for nonlinear DSGE models. We provide theoretical and empirical examples to compare our method with the only existing method called generalized forecast error variance decomposition (GFEVD). For the empirical example, we use a standard dynamic New Keynesian (DNK) model with an occasionally-binding zero lower bound on nominal interest rates (ZLB). We find that the results from the two methods are very different where shocks are multiplicative or interacted in nonlinear models. We

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recommend that when working with nonlinear DSGE models researchers should use the total variance method in order to see the importance of indirect variance contributions and to quantify correctly the relative variance contribution of each structural shock. We also find from the empirical example that the supply shock becomes significantly less important relative to the demand shock in explaining the volatility of economic variables, especially GDP, at a short horizon when the economy is in a deep recession with a binding ZLB.

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1 Introduction

Nonlinear or global solution methods for nonlinear DSGE models such as dynamic New Keynesian (DNK) models have become increasingly popular recently due to an occasionally-binding zero lower bound on nominal interest rates (ZLB) and other constraints.\textsuperscript{1} With global solution methods, policy functions are nonlinear and results are more accurate; see Fernandez-Villaverde et al. (2015) for example.\textsuperscript{2} However, computing and analyzing forecast error variance decomposition (FEVD) become more complicated because impulse responses and variance decomposition are not only state dependent but also shock and composition dependent (Koop et al. (1996)).\textsuperscript{3}

Although the generalized impulse response function (GIRF) proposed by Koop et al. (1996) has been used more frequently to analyze dynamic responses for nonlinear DSGE models, variance decomposition analyses are very limited. Nevertheless, such analyses often provide important economic insights. For example, Gourio and Ngo (2016) show that the ZLB constraint would alter the response of economic variables to shocks, leading to change in relative importance of shock to risk premia and business cycles.

One important reason for the lack of variance decomposition analyses for nonlinear DSGE models is the absence of a comprehensive framework and/or efficient methods to implement them. According to our knowledge, the only existing method developed and used in economics is the generalized forecast error variance decomposition (GFEVD)

\textsuperscript{1}In a linearized version of a nonlinear DSGE model, Taylor rules are still nonlinear due to the ZLB. Thus, the ”linearized” model is actually nonlinear.

\textsuperscript{2}An incomplete list of papers using nonlinear models with a ZLB constraint includes Wolman (2005), Nakata (2016), Ngo (2014b), and Richter et al. (2014).

\textsuperscript{3}As explained above, a linearized version of nonlinear DSGE models is still nonlinear due to the ZLB. Thus, resulting policy functions are nonlinear and the traditional variance decomposition method for linear vector autoregression (VAR) is not applicable.
method that is proposed by Pesaran and Shin (1998) and recently modified by Lanne and Nyberg (2016) to ensure that relative variance contributions by structural shocks are added up to one hundred percent.

In this paper, we aim at filling the gap by proposing a "new" method called the total variance method to implement FEVD for nonlinear DSGE models. This method is an application of the law of total variance. In addition, we compare the method with the only existing GFEVD method using theoretical and empirical examples and to answer the question "what method practitioner should use". For the empirical example, we apply these methods to a calibrated standard DNK model with an occasionally-binding ZLB. More specifically, we study the relative importance of structural demand and supply shocks to business cycles in normal times when the ZLB does not bind and in the Great Recession when the ZLB binds.

Our main contribution is threefold. First, we provide a comprehensive framework to compute and analyze FEVD for nonlinear DSGE models. While the GFEVD method is not new, our method is "novel" because, although it has been used in engineering and statistics it is "new" to economics, or at least not well documented. Second, we provide both theoretical and empirical examples to illustrate differences between the two methods. Finally, we provide our algorithms for the methods.

We find that these two methods generate very different results when structural shocks are interacted in the policy functions of a nonlinear model. Therefore, we recommend that when working with nonlinear DSGE models researchers should use the total variance method in order to see the importance of indirect variance contributions and to quantify correctly the relative variance contribution for each structural shock. Using our decomposition method, we quantify the relative contribution of supply and demand shocks in the Great Recession with a binding ZLB. We find that the supply shock becomes
significantly less important relative to the demand shock in explaining the volatility of economic variables, especially at a short horizon. This occurs because the magnitude of response of GDP to a technology shock becomes much smaller relative to the magnitude of response of GDP to a preference shock.

This paper is related to the literature that investigates impulse responses and FEVD for nonlinear vector autoregression (VAR) models. While the literature that studies impulse responses for nonlinear VAR models was dated back to the seminal work by Koop et al. (1996), the research in FEVD for nonlinear models has been limited. The only papers related to ours are Pesaran and Shin (1998) and Lanne and Nyberg (2016). Pesaran and Shin (1998) proposed the GFEVD method, while Lanne and Nyberg (2016) modify the GFEVD method to ensure that all the relative variance contributions of structural shocks add up to one hundred percent. Lanne and Nyberg’s GFEVD method is very similar to the traditional forecast error variance decomposition for a linear VAR model. The difference is that they use generalized impulse responses instead of traditional impulse response in their formula. Lanne and Nyberg (2016) apply this method to nonlinear multivariate models with additive errors.\footnote{In terms of nonlinear solution methods, our paper is most closely related to Fernandez-Villaverde et al. (2015), Ngo (2014b), Gust et al. (2012), and Aruoba and Schorfheide (2013). All these papers use global projection methods to approximate policy functions in DNK models with a ZLB constraint.}

The paper is organized as follows. Our new method to compute FEVD for a nonlinear DSGE model is introduced in Section 2. We also summarize the GFEVD method in this section. In addition, we provide a simple theoretical example to illustrate differences among these methods. Section 3 applies these methods to a standard DNK model with an occasionally binding ZLB to study the relative importance of supply and demand shocks in explaining business cycles. We conclude in Section 4. Our algorithms, data
description, and solution method are included in the appendix.

2 Variance decomposition methods for nonlinear DSGE models: Theoretical framework

The system of nonlinear equations governing equilibria for any nonlinear DSGE model can be solved numerically, and resulting nonlinear policy functions can be cast in a state space model as following:

\[
Y_t = f(S_{t-1}, \epsilon_t), \tag{1}
\]

where \( f(\cdot) : \mathbb{R}^{n+2k} \rightarrow \mathbb{R}^n \) is a known nonlinear function; \( Y_t \) is an \( n \times 1 \) vector of endogenous variables; \( S_{t-1} = (Y_{t-1}; s_{t-1}) \) is the vector of state variables; \( Y_{t-1} \) is a \( n \times 1 \) vector of endogenous state variables; \( s_t \) is a \( k \times 1 \) vector of exogenous state variables that has the following motion equation:

\[
s_t = As_{t-1} + \epsilon_t; \tag{2}
\]

\( A \) is a known \( k \times k \) matrix; \( \epsilon_t \) is a \( k \times 1 \) vector of orthogonal shocks with a known diagonal variance-covariance matrix \( \Sigma_\epsilon \) and mean \( 0_{k \times 1} \).

This type of nonlinear policy function nests the linear VAR model studied in Koop et al. (1996) and nonlinear multivariate model with additive errors considered in Lanne and Nyberg (2016). The impulse response functions (IRF) based on this nonlinear policy function are shock, history, and composition dependent. It means that the response of \( Y_{t+h} \) under any single shock \( \epsilon_j^t \) for \( j \in \{1, \ldots, k\} \) may depend on the state of the economy at time \( t-1 \) (\( S_{t-1} \)), the size and sign of \( \epsilon_j^t \), and the signs and sizes of all the shocks from
time $t$ to $t + h$, $\epsilon_{t:t+h}$.\textsuperscript{5} Therefore, Koop et al. (1996) proposed that we use generalized impulse response function (GIRF), instead of traditional impulse response function. In addition, due to the shock, history, and composition dependence, we cannot compute FEVD using the traditional way for a linear VAR model.

To provide the reader with some background on variance decomposition for nonlinear multivariate models, we first summarize the only existing method developed and used in economics, which is the GFEVD method. We then introduce our ”new” method called the total variance method. This method is ”new” because, although it has been used widely in engineering and statistics (Harris and Yu (2012)), it has not been formally introduced and well documented in economics, especially for a nonlinear DSGE framework.

### 2.1 The GFEVD method

The GFEVD method is the only existing method formally documented to implement FEVD analyses (Lanne and Nyberg (2016)). It is initially developed by Pesaran and Shin (1998) for a linear VAR model. Lanne and Nyberg (2016) modify this method and apply to nonlinear multivariate models with additive errors. In principle, we could apply this method to study variance decomposition for nonlinear DSGE models after obtaining nonlinear policy function (1). In particular, let $\epsilon_{t}^{-j}$ denote the vector of all orthogonal shocks excluding $j$-th element.\textsuperscript{6} So $\epsilon_{t}^{-j}$ is a $(k-1) \times 1$ vector. In addition, let $Y_t$ denote a scalar component of vector variable $Y_t$. From Koop et al. (1996), the GIRF for variable

\textsuperscript{5}Throughout this paper, we use $X$ to denote a component of vector $X$. For example, $\epsilon$ denotes a single shock, which is a component of the vector of structural shocks $\epsilon$.

\textsuperscript{6}Note that all the variance decomposition methods used in this paper are based on simulation after obtaining a numerical policy function. So, a reduction of dimension from $k$ to $(k-1)$ will not violate any dimension conformity.
$Y_{t+h}$ at horizon $h$ under a shock of magnitude $\delta_j$ to $\epsilon^j_t$, conditional on state $S_t$ is:

$$GI_t^j (h|\epsilon_{t+1}^j = \delta_j) = \mathbb{E}_t [Y_{t+h}|\epsilon_{t+1}^j = \delta_j] - \mathbb{E}_t [Y_{t+h}], \quad (3)$$

where the first expectation is taken over the distribution of $\epsilon_{t+1}^j$ and $\epsilon_{t+2,t+h}$; the second expectation is taken over the distribution of $\epsilon_{t+1:t+h}$. Following Lanne and Nyberg (2016), we can compute the FEVD of shock $j$ conditional on state $S_t$:

$$\lambda_t^j (h|\epsilon_{t+1}^j = \delta_j) = \frac{\sum_{l=1}^h (GI_t^j (l|\epsilon_{t+1}^j = \delta_j))^2}{\sum_{i=1}^m \sum_{l=1}^h (GI_t^i (l|\epsilon_{t+1}^i = \delta_i))^2}. \quad (4)$$

This formula is similar the traditional forecast error variance decomposition for a linear VAR model with orthogonal shocks. The difference is that this formula uses generalized impulse responses instead of traditional impulse responses. Pesaran and Shin (1998) suggest that we use scaled GIRFs by setting $\delta_j$ to the standard deviation of shock $j$. In addition, Lanne and Nyberg (2016) suggest that we should integrate over the shock size to compute the variance decomposition attributed to shock $j$ at horizon $h$ conditional on state $S_t$:

$$\lambda_t^j (h) = \mathbb{E} [\lambda_t^j (h|\epsilon_{t+1}^j)], \quad (5)$$

where the expectation is taken over the distribution of shock size $\delta_j$, which is also the distribution of $\epsilon_{t+1}^j$.

### 2.2 The total variance method

Like most variance decompositions, the total variance method is based on the law of total variance (or the law of iterated expectations for variance). Let $\mathbb{E}_t [Y_{t+h}|\epsilon_{t+1:t+h}^j]$ and
\[ \mathbb{V}_t \left[ Y_{t+h} \big| \epsilon_{t+1:t+h}^{-j} \right] \] denote the expectation and the variance of \( Y_{t+h} \) over the distribution of shock \( j \) from time \( t + 1 \) to \( t + h \), \( \epsilon_{t+1:t+h}^j \), conditional on state \( S_t \) and a path of the other shocks, \( \epsilon_{t+1:t+h}^{-j} \).

According the law of total variance, the variance of \( Y_{t+h} \) conditional on state \( S_t \)

\[ \mathbb{V}_t \left[ Y_{t+h} \right] = \mathbb{E}_t \left[ \mathbb{V}_t \left[ Y_{t+h} \big| \epsilon_{t+1:t+h}^{-j} \right] \right] + \mathbb{V}_t \left[ \mathbb{E}_t \left[ Y_{t+h} \big| \epsilon_{t+1:t+h}^{-j} \right] \right]. \] (6)

The first term in the RHS of (6) is the variance contribution of shock \( j \) to the total variance of \( Y_{t+h} \) averaged across all possible paths of the other shocks. This term comprises the direct effect of shock \( j \) together with interaction effects of shock \( j \) with other shocks. We denote this term as

\[ \mathbb{V}_t^j \left[ Y_{t+h} \right] \overset{\Delta}{=} \mathbb{E}_t \left[ \mathbb{V}_t \left[ Y_{t+h} \big| \epsilon_{t+1:t+h}^{-j} \right] \right]. \] (7)

The second term in the RHS of (6) is the residual part of the total variance due to all the other shocks. This term is also interpreted as the direct contribution of all the shocks other than shock \( j \).

We then compute forecast error variance decomposition (FEVD) for shock \( j \) at horizon \( h \) conditional on state \( S_t \) as a fraction of the total variance in \( Y_{t+h} \) due to shock \( j \):

\[ \lambda_j^t (h) \overset{\Delta}{=} \frac{\mathbb{V}_t^j \left[ Y_{t+h} \right]}{\mathbb{V}_t \left[ Y_{t+h} \right]} = 1 - \frac{\mathbb{V}_t \left[ \mathbb{E}_t \left[ Y_{t+h} \big| \epsilon_{t+1:t+h}^{-j} \right] \right]}{\mathbb{V}_t \left[ Y_{t+h} \right]} . \] (8)

It follows that FEVD (8) recognizes both the direct contribution of shock \( j \) to the total variance and indirect contributions stemming from the interactions of shock \( j \) and the other shocks. In contrast, the GFEVD method accounts for the direct contribution only.\(^7\)

\(^7\)We thank an anonymous referee for this excellent point. In a subsection below, we provide a
In the multivariate models studied by Lanne and Nyberg (2016), shocks are additive so the indirect contribution of shock interactions is zero. This method would result the same result as their GFEVD method. However, for nonlinear policy functions as studied in this paper, where shocks are multiplicative and interacted, we should quantify the variance attributed to shock $j$ both directly and indirectly. Therefore, the total variance method, as indicated by FEVD (8), is a more reasonable.

The second equality in (8) follows from (6) and emphasizes that the fractional contributions of shock $j$ and all other shocks are added up to unity. As fractions of the total variance, the FEVD’s of alternative shocks can be directly compared to each other. In general, the sum of the FEVD’s is not equal to one. To archive the adding-up property, we define the normalized FEVD as

$$\tilde{\lambda}^j_t(h) \triangleq \frac{\lambda^j_t(h)}{\sum_{i=1}^{m} \lambda^i_t(h)}.$$  \hspace{1cm} (9)

### 2.3 A theoretical example

In this section, we provide a theoretical example to show that: (i) the GFEVD method accounts for the direct contribution only; (ii) the total variance method as in equation (9) recognizes both the direct contribution of shock $j$ and indirect contribution stemming from interactions between shock $j$ and other shocks.

Consider a theoretical nonlinear policy function in the following form:

$$y_t = \rho y_{t-1} + e_{1,t} + e_{2,t} + \alpha_1 e_{1,t} e_{2,t} + \alpha_2 e_{1,t} e_{3,t},$$ \hspace{1cm} (10)

where $e_{1,t}, e_{2,t}, e_{3,t}$ are serially and contemporaneously structural shocks, which are in-theoretical example to illustrate this point.
dependently and identically distributed with mean zero and variances: $\sigma_1^2$, $\sigma_2^2$, and $\sigma_3^2$, respectively.

For the sake of illustration, let us consider a one-step-ahead forecast, i.e. $h = 1$. It is not difficult to see that:

\[
\begin{align*}
\mathbb{V}_t[y_{t+1}] & = \sigma_1^2 + \sigma_2^2 + \alpha_1^2 \sigma_1^2 \sigma_2^2 + \alpha_2^2 \sigma_1^2 \sigma_3^2, \\
\mathbb{V}_t^1[y_{t+1}] & = \sigma_1^2 + \alpha_1^2 \sigma_1^2 \sigma_2^2 + \alpha_1^2 \sigma_1^2 \sigma_3^2, \\
\mathbb{V}_t^2[y_{t+1}] & = \sigma_2^2 + \alpha_1^2 \sigma_1^2 \sigma_2^2, \\
\mathbb{V}_t^3[y_{t+1}] & = \alpha_2^2 \sigma_1^2 \sigma_3^2.
\end{align*}
\]

Therefore, from equation (9), the FEVD based on the law of total variance is as following:

\[
\begin{align*}
\lambda_1^1(1) & = \frac{\sigma_1^2 + \alpha_1^2 \sigma_1^2 \sigma_2^2 + \alpha_1^2 \sigma_1^2 \sigma_3^2}{\sigma_1^2 + \sigma_2^2 + 2\alpha_1^2 \sigma_1^2 \sigma_2^2 + 2\alpha_2^2 \sigma_1^2 \sigma_3^2}, \\
\lambda_2^1(1) & = \frac{\sigma_1^2 + \alpha_1^2 \sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2 + 2\alpha_1^2 \sigma_1^2 \sigma_2^2 + 2\alpha_2^2 \sigma_1^2 \sigma_3^2}, \\
\lambda_3^1(1) & = \frac{\alpha_2^2 \sigma_1^2 \sigma_3^2}{\sigma_1^2 + \sigma_2^2 + 2\alpha_1^2 \sigma_1^2 \sigma_2^2 + 2\alpha_2^2 \sigma_1^2 \sigma_3^2}.
\end{align*}
\]

The generalized impulse response function (GIRF) is:

\[
GI_t^j(1|e_{j,t+1} = \delta_j) = \mathbb{E}_t[y_{t+1}|e_{j,t+1} = \delta_j] - \mathbb{E}_t[y_{t+1}].
\]

Taking the expectation over $e_{j,t+1}$ of the squared GIRF gives

\[
\mathbb{E}_t\left[\left(GI_t^j(1|e_{j,t+1})\right)^2\right] = \mathbb{E}_t\left[\left(\mathbb{E}_t[y_{t+1}|e_{j,t+1}] - \mathbb{E}_t[y_{t+1}]\right)^2\right]
= \mathbb{V}_t[\mathbb{E}_t[y_{t+1}|e_{j,t+1}]].
\]
For the model (10)

\[ \forall_t [E_t [y_{t+1} | e_{j,t+1}]] = \sigma_j^2 \text{ for } j = 1, 2, \]

\[ = 0 \text{ for } j = 3. \]

Thus, the FEVD based on the GIRF produces:

\[ \lambda_t^1(1) = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}, \quad (14) \]
\[ \lambda_t^2(1) = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \quad (15) \]
\[ \lambda_t^3(1) = 0. \quad (16) \]

Apparently, FEVD’s given by (11)-(13) and (14)-(16) coincide in the case of no interaction between shocks, i.e. \( \alpha_1 = \alpha_2 = 0 \). It is to note that FEVD definition (14)-(16) deviates from those in GFEVD method due to the nonlinearity of the GFEVD formula. In particular, to derive (14)-(16), we take expectation of the squared GIRF and then compute the ratio, while the GFEVD method, as indicated by equations (4) and (5), suggests averaging the ratio of the squared GIRF’s.

Finally, in this example, it is straightforward to show that that the variance decomposition using the two methods remains unchanged for the horizon \( h > 1 \).

3 An empirical example

In this section, we use a conventional dynamic New Keynesian (DNK) model with an occasionally binding ZLB to illustrate differences between the two variance decomposition

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\( ^8 \)Our simulations based on the algorithms provided in the paper confirmed the theoretical results. This serves as a sanity check for our algorithms.
methods.\textsuperscript{9}

### 3.1 Model

Our model consists of a continuum of identical households, a continuum of identical competitive final goods producers, a continuum of monopolistically competitive intermediate goods producers, and a government (monetary and fiscal authorities).

#### 3.1.1 Households

The representative household maximizes his expected discounted utility

\[
E_1 \left\{ \sum_{t=1}^{\infty} (\Pi^t_{j=0} \beta_j) \left( \frac{C_t^{1-\gamma}}{1-\gamma} - \chi \frac{N_t^{1+\eta}}{1+\eta} \right) \right\}
\]

subject to the budget constraint

\[
P_t C_t + (1 + i_t)^{-1} B_t = W_t N_t + B_{t-1} + \Pi_t + T_t,
\]

where \( C_t \) is consumption of final goods, \( i_t \) is the nominal interest rate, \( B_t \) denotes one-period bond holdings, \( N_t \) is labor, \( W_t \) is the nominal wage rate, \( \Pi_t \) is the profit income, \( T_t \) is the lump-sum tax, and \( \beta_t \) denotes the preference shock. We assume that \( \beta_t \) follows

\textsuperscript{9}The main contribution of this paper is to provide a "new" method to conduct variance decomposition, to compare this method with the existing method, and to answer the question "which method practitioners should use". So, we keep our model standard. The key insight about differences between the methods will not change if we use a larger model. However, the empirical results are not necessarily generalized to un-nested larger models. Using a larger scale DSGE model as in Gust et al. (2017) to study the relative importance of structural shocks above and at the ZLB is in our future research agenda.
an AR(1) process

\[ \ln(\beta_t) = (1 - \rho_\beta) \ln \beta + \rho_\beta \ln(\beta_{t-1}) + \epsilon_{\beta t}, \quad \beta_0 = 1 \] (19)

where \( \rho_\beta \in (0, 1) \) is the persistence of the preference shock and \( \epsilon_{\beta t} \) is the innovation of the preference shock with mean 0 and variance \( \sigma_\beta^2 \). The preference shock is a reduced form of more realistic forces that can drive the nominal interest rate to the ZLB.\(^{10}\)

The first-order conditions for the household optimization problem are given by

\[ \chi N_t^\gamma C_t^\gamma = w_t, \] (20)

and

\[ \mathbb{E}_t \left[ M_{t,t+1} \left( \frac{1 + i_t}{1 + \pi_{t+1}} \right) \right] = 1, \] (21)

where \( w_t = W_t/P_t \) is the real wage, \( \pi_t = P_t/P_{t-1} - 1 \) is the inflation rate, and the stochastic discount factor is given by

\[ M_{t,t+1} = \beta_t \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma}. \] (22)

\(^{10}\)This setting is very common in the ZLB literature; for example, see Eggertsson and Woodford (2003), Nakata (2016) and Ngo (2014b) among others. Another way to make the ZLB binding is to introduce a deleveraging shock as in Eggertsson and Krugman (2012), Guerrieri and Lorenzoni (2011), and Ngo (2015).
3.1.2 Final goods producers

To produce final goods, final goods producers buy and aggregate a variety of intermediate goods indexed by \( i \in [0, 1] \) using a CES technology

\[
Y_t = \left( \int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} \, di \right)^{\frac{1}{\epsilon-1}},
\]

where \( \epsilon \) is the elasticity of substitution among intermediate goods. The profit maximization problem is given by

\[
\max \ P_t Y_t - \int_0^1 P_t(i) Y_t(i) \, di,
\]

where \( P_t(i) \) and \( Y_t(i) \) are the price and quantity of intermediate good \( i \). Profit maximization and the zero-profit condition give the demand for intermediate good \( i \)

\[
Y_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\epsilon} Y_t, \quad (23)
\]

and the aggregate price level

\[
P_t = \left( \int P_t(i)^{1-\epsilon} \, di \right)^{\frac{1}{1-\epsilon}}. \quad (24)
\]

3.1.3 Intermediate goods producers

There is a unit mass of intermediate goods producers on \([0, 1]\) that are monopolistic competitors. Suppose that each intermediate good \( i \in [0, 1] \) is produced by one producer using the linear technology

\[
Y_t(i) = A_t N_t(i), \quad (25)
\]
where \( N_t(i) \) is labor input and \( A_t \) denotes the supply shock that follows an AR(1) process:

\[
\ln A_t = \rho_A \ln A_{t-1} + \epsilon_{At},
\]

(26)

where \( \rho_A \in (0, 1) \) is the persistence parameter and \( \epsilon_{At} \) is the innovation with mean 0 and variance \( \sigma_{A}^2 \). Cost minimization implies that each firm faces the same real marginal cost divided by productivity:

\[
mc_t = mc_t (i) = \frac{w_t}{A_t}.
\]

(27)

#### 3.1.4 Price setting

Following Rotemberg (1982), we assume that each intermediate goods firm \( i \) faces costs of adjusting prices in terms of final goods. In this paper, we use a quadratic adjustment cost function, which was proposed by Ireland (1997) and which is one of the most common functions used in the ZLB literature:

\[
\frac{\varphi}{2} \left( \frac{P_t (i)}{P_{t-1} (i)} - 1 \right)^2 Y_t,
\]

where \( \varphi \) is the adjustment cost parameter which determines the degree of nominal price rigidity.\(^{11}\) The problem of firm \( i \) is given by

\[
\max_{\{P_t(i)\}} \mathbb{E}_t \sum_{j=0}^{\infty} \left\{ M_{t,t+j} \left[ \left( \frac{P_{t+j} (i)}{P_{t+j-1} (i)} - mc_t \right) Y_{t+j} (i) - \frac{\varphi}{2} \left( \frac{P_{t+j} (i)}{P_{t+j-1} (i)} - 1 \right)^2 Y_{t+j} \right] \right\}
\]

(28)

\(^{11}\)A non-exhaustive list of studies that use this price adjustment cost at the ZLB includes Nakata (2011), Aruoba and Schorfheide (2013), Miao and Ngo (2018), and Liu et al. (2018). It would also be interesting to compare the time-dependent Calvo price setting to another state-dependent price setting as in Dotsey et al. (1999) and Ngo (2014a) at the ZLB.
subject to its demand (23). In a symmetric equilibrium, all firms will choose the same price and produce the same quantity, i.e., \( P_t(i) = P_t \) and \( Y_t(i) = Y_t \). The optimal pricing rule then implies that

\[
\left(1 - \epsilon + \epsilon \frac{w_t}{A_t} - \varphi \pi_t (1 + \pi_t)\right) Y_t + \varphi \mathbb{E}_t [M_{t+1} \pi_{t+1} (1 + \pi_{t+1}) Y_{t+1}] = 0. \tag{29}
\]

### 3.1.5 Monetary and fiscal policies

The central bank conducts monetary policy by setting the interest rate using a simple Taylor rule, subject to the ZLB condition,

\[
\frac{1 + i_t}{1 + i} = \max \left\{ \left( \frac{GDP_t}{GDP} \right)^{\phi_y} \left( \frac{1 + \pi_t}{1 + \pi} \right)^{\phi_\pi} \frac{1}{1 + i} \right\}, \tag{30}
\]

where the gross domestic product

\[
GDP_t = C_t + G_t; \tag{31}
\]

\( G_t = 0.2GDP_t \); and \( GDP, \pi, \) and \( i \) denote the steady state GDP level, the targeted inflation rate, and the steady-state nominal interest rate, respectively.\(^{12}\)

### 3.2 Equilibrium systems

With the Rotemberg price setting, aggregate output satisfies

\[
Y_t = A_t N_t, \tag{32}
\]

\(^{12}\)Some researchers use flexible-price output as an output target in the Taylor rule, and some researchers also include lagged interest rates. These alternative specifications will not change our key insights.
and the resource constraint is given by

\[ GDP_t + \frac{\varphi}{2} \pi_t^2 Y_t = Y_t. \]  

(33)

The equilibrium system for the Rotemberg model consists of a system of six nonlinear difference equations (20), (21), (29), (30), (31), (32), and (33) for seven variables \( w_t, C_t, \)

\( i_t, \pi_t, N_t, Y_t, \) and \( GDP_t. \)

### 3.3 Calibration

We calibrate the primitive parameters of the model based on the existing literature.\(^{13}\) The quarterly subjective discount factor is set at \( \beta = 0.995 \) such that the annual real interest rate is 2%, as in Christiano et al. (2011) and Boneva et al. (2016). The constant relative risk aversion parameter is \( \gamma = 1 \), corresponding to a log utility function with respect to consumption. This utility function is commonly used in the business cycles literature. The labor supply elasticity with respect to wages is set at \( \eta = 1 \), as in Christiano et al. (2011). The value of \( \chi \) is calibrated to obtain the steady state fraction of working hours of \( 1/3 \). The elasticity of substitution among differentiated intermediate goods is \( \epsilon = 7.66 \), corresponding to a 15% net markup. This value is also popular in the literature (e.g. Boneva et al. (2016)).

We set the price adjustment cost parameter in the Rotemberg model \( \varphi = 495.8 \), as in Boneva et al. (2016). This value, together with the other parameters, implies that the slope of the Phillips curve is 0.0269, which is within the range estimated by Ball and Mazumder (2011) for the U.S. using the 1985:q1-2007:q4 data.

\(^{13}\)Another way to assign values to the primitive parameters of the model is to estimate the model using particle filtering and let the data speak out. However, even in this case, many parameters still need calibrating; see Richter and Throckmorton (2016) and Gust et al. (2017) for more information.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Values</th>
</tr>
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<tr>
<td>(\beta)</td>
<td>Quarterly discount factor</td>
<td>0.995</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>CRRA parameter</td>
<td>1</td>
</tr>
<tr>
<td>(\eta)</td>
<td>Inverse labor supply elasticity</td>
<td>1</td>
</tr>
<tr>
<td>(\epsilon)</td>
<td>Monopoly power</td>
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</tr>
<tr>
<td>(\varphi)</td>
<td>Price adjustment cost parameter in the Rotemberg model</td>
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<tr>
<td>(\pi)</td>
<td>Inflation target</td>
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<tr>
<td>(\phi_\pi)</td>
<td>Weight of inflation target in the Taylor rule</td>
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</tr>
<tr>
<td>(\phi_y)</td>
<td>Weight of output target in the Taylor rule</td>
<td>(\frac{0.5}{4})</td>
</tr>
<tr>
<td>(\sigma_\beta)</td>
<td>Standard deviation of preference innovations</td>
<td>(\frac{0.13}{100})</td>
</tr>
<tr>
<td>(\rho_\beta)</td>
<td>AR-coefficient of preference shocks</td>
<td>0.85</td>
</tr>
<tr>
<td>(\sigma_A)</td>
<td>Standard deviation of technology innovations</td>
<td>(\frac{0.25}{100})</td>
</tr>
<tr>
<td>(\rho_A)</td>
<td>AR-coefficient of government spending shocks</td>
<td>0.90</td>
</tr>
</tbody>
</table>

Table 1: Parameter calibration.
We set the parameters in the Taylor rule at $\phi_x = 1.75$ and $\phi_y = 0.25$, which are close to the estimates by Gust et al. (2017). The inflation target is set to be zero.$^{14}$

Following Fernandez-Villaverde et al. (2015), we set the persistence of technology shock $\rho_A = 0.9$ and the standard deviation for the shock innovations $\sigma_A = \frac{0.25}{100}$. They argue that this technology shock specification explains the U.S. data reasonably well for the past two decades.

The most important parameters left to calibrate are those regarding the preference shock specification. Following Gust et al. (2017), we set the persistence of preference shock $\rho_\beta = 0.85$. We set the standard deviation for preference innovations $\sigma_\beta = \frac{0.13}{100}$ so that the unconditional probability of hitting the ZLB is about 17%, which is in the range implied by the recent U.S. data. See Ball (2013) and Ngo (2018) for more information.

### 3.4 Solution and FEVD

In terms of a numerical solution, we use projection methods that are similar to Ngo (2014b). In particular, we solve for policy function using a finite element method called the linear spline interpolation (see Judd (1998) and Miranda and Fackler (2002)).$^{15}$ The policy can be cast in the form of equation (1), as described in Section 2. In particular, let $Y = (i,C,N,Y,GDP,w,\pi)'$, where $i, C, N, Y, GDP, w,$ and $\pi$ denote the nominal interest rate, consumption, labor, output, GDP, the wage rate, and the inflation rate, respectively. Then,

$^{14}$If the steady-state inflation rate is not zero, we can rewrite the model to allow for inflation indexation in firms’ price setting and for inflation gap stabilization in the Taylor rule. This transformation will not alter the solution of the model and the variance decomposition results.

$^{15}$More detail about the solution method can be found in Appendix B.
\[ Y_t = f(S_{t-1}, \epsilon_t), \quad (34) \]

where \( f(\cdot) : \mathbb{R}^{11} \to \mathbb{R}^7 \) is a known nonlinear function that we have solved; \( S_{t-1} = (Y_{t-1}; s_{t-1}) \) is the vector of endogenous and exogenous state variables; the vector of exogenous state variables \( s_t = (A_{t-1}, \beta_{t-1})' \) has the following transition equation:

\[ s_t = As_{t-1} + \epsilon_t; \quad (35) \]

\[
A = \begin{pmatrix}
\rho_A & 0 \\
0 & \rho_\beta
\end{pmatrix}; \quad \epsilon_t = (\epsilon_{A,t}, \epsilon_{\beta,t})'
\]

is the 2 \times 1 vector of orthogonal supply (technology) and demand (preference) shocks with a known diagonal variance-covariance matrix \( \Sigma_\epsilon = \begin{pmatrix}
\sigma_A^2 & 0 \\
0 & \sigma_\beta^2
\end{pmatrix} \) and mean \( 0_{2 \times 1}. \)

To implement FEVD, we follow closely our algorithms presented in Appendix A. To be consistent with the notation in Section 2, let us use \( Y \) to denote any single component of \( Y \). It is important to note that the total variance and the GFEVD methods are merely based on Monte Carlo simulations.

In this standard DNK model, there is not any endogenous state variable \( Y_{t-1} \). Therefore, the coefficients associated with these endogenous variables in the policy function are zero. However, we keep using \( Y_{t-1} \) to ensure the generality of our equations and algorithms.

It is tempting for the reader to ask us to work with a linearized version of this model and compare variance decomposition results from both linearized and fully-nonlinear models. However, Fernandez-Villaverde et al. (2015) and Ngo (2014b) show that linearized models do not produce an accurate policy function at the ZLB. In order to implement FEVD analyses correctly and to provide a meaningful comparison among the methods, working with an accurate nonlinear policy function that comes from a fully-nonlinear model is a necessary condition.
3.5 Results

In this section we compute and report FEVD for two states: the steady state when the ZLB does not bind and the Great Recession when the ZLB binds, i.e. the second quarter of 2009.

3.5.1 Filtering the Great Recession state

The question is how we obtain the state of the economy at the trough of the Great Recession. To this end, we filter the unobserved state that consists of preference and technology (TFP) using US data and a particle filter.

The data are quarterly and include real GDP, inflation based on the GDP deflator, and the effective federal funds rate spanning from 1982Q1 to 2018Q2. The matrix of observables is:

\[
\mathbf{x}_{data}^t = \left[ \log \left( \frac{GDP_t}{DEF_t/DEF_{t-1}} \right), \log \left( \frac{1 + FFR_t/100}{4} \right) \right]
\]  

(36)

where \( GDP \) is the real GDP, \( DEF \) is the GDP deflator, \( FFR \) is the effective federal funds rate. More detailed description of the data can be found in Appendix A.

We first Hodrick-Prescott (HP) filter our data to keep business cycle components only. This is because our model is just the standard New Keynesian model that does not explicitly take into account trends for real GDP, inflation, and interest rates. The filtered values multiplied by 100 can be interpreted as a percent deviation from the trend.

The link between the model and the data is:

\[
\tilde{x}_{data}^t = \tilde{x}_{model}^t + \zeta_t,
\]

(37)

22
where $\tilde{x}_t^{data}$ is the HP filtered component of $x_t^{data}$ and

$$
\tilde{x}_t^{model} = \left[ \log \left( \frac{Y_t^{gdp}}{\bar{Y}^{gdp}} \right), \log \left( \frac{\Pi_t}{\bar{\Pi}} \right), \log \left( \frac{R_t}{\bar{R}} \right) \right]
$$

(38)

and $\zeta \sim N(0, \Sigma)$ is the vector of measurement errors and $\Sigma = \text{diag} \left( \sigma_{\zeta,y}^2, \sigma_{\zeta,\pi}^2, \sigma_{\zeta,R}^2 \right)$. We set the variance of each measurement error to 10% of the variance of each observable. Different values would have a little effect on the filtering result. Note $Y_t^{gdp} \equiv C + G$ is the model’s GDP. Both $\tilde{x}_t^{data}$ and $\tilde{x}_t^{model}$ are percent deviation from trend/steady state.

Figure 1 shows the actual and model-based percent deviations from trend/steady state for real GDP, inflation, and interest rates over the period from 1982Q1 to 2018Q2, while Figure 2 presents the filtered state for the preference and the technology. At the trough of the Great Recession (2009Q2), the subjective discount factor increased substantially and the economy was at a state with highly positive technology. The combined effect leads to a binding ZLB with a deep recession and deflation. In particular, real GDP declines by 2.7% and inflation drops by approximately 2.5% from their trends. This means the output gap is $-2.7\%$ and the inflation rate is $-0.5\%$ per annum. In addition, the median ZLB duration is 4 quarters starting from the Great Recession state.$^{18}$ The characteristics of the Great Recession state are in line with US data.

### 3.5.2 Variance decompositions at steady state

Table 2 shows the variance decompositions for GDP and inflation at different horizons starting from steady state, based on the total variance method and the GFEVD method. We use the numbers of simulations: $N_m = N_t = 5,000$ for these methods. See Section

$^{18}$Based on our simulations of one millions quarters, there are many episodes where the ZLB binds more than 30 periods consecutively. This is not unusual given the fact that we calibrated the shocks such that the probability of hitting the ZLB is about 17% and the shocks are persistent.
Figure 1: Actual and model-based percent deviation from trend/steady state for real GDP, inflation, and interest rates from 1982Q1 to 2018Q2. The shaded bars represent recessions.
Figure 2: Filtered preference and TFP from 1982Q1 to 2018Q2. The shaded bars represent recessions.
3 for more detail. Increasing the numbers of simulations further would not change our main results.

Given our parameter calibration, the total variance method shows that at steady state the technology shock contributes 12.62% and 27.43% to the forecast error variance of GDP and inflation at 1-period horizon, respectively. The contribution by technology shocks increases at a longer horizon. In particular, at a 20-period horizon the technology shock’s contributions to GDP and inflation volatility increase to 13.00% and 34.89%, respectively. Based on the parameterization, the preference shock dominates the technology shock in explaining the volatility of GDP and inflation at all horizons.

It is very important to note that, the GFEVD method produces a result different from that of the total variance method, especially at short horizons. For example, at a 1-period horizon the GFEVD method generates around 16.61% and 34.69% contribution of technology to GDP and inflation, respectively. However, the values from the GFEVD are only 12.62% and 27.43%. This means that the indirect contribution of technology shocks to GDP and inflation is smaller than that of preference shocks. In other words, higher-order effects stemming from shock interactions plays a significant role in quantifying relative variance contribution of each shock, as seen in the theoretical part of section 2.

3.5.3 Variance decompositions at the Great Recession

Table 2 also shows the variance decompositions for GDP and inflation at different horizons starting from the Great Recession state with binding ZLB, based on the total variance method and the GFEVD method.

It is interesting to see that the relative importance of technology and preference in explaining the volatility of GDP and inflation change substantially in this Great Recession case, compared to the case of steady state, especially at a short horizon when
### Panel A. GDP

<table>
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<tr>
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<th>TFP Pref.</th>
<th>TFP Pref.</th>
<th>TFP Pref.</th>
<th>TFP Pref.</th>
<th>TFP Pref.</th>
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<td>8</td>
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### Panel B. Inflation

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</thead>
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<td>0.7691</td>
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<td>3</td>
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<td>4</td>
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<td>0.7802</td>
<td>0.2677</td>
<td>0.7323</td>
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Table 2: Variance decompositions at steady state and at the Great Recession. For the total variance method and the GFEVD method, we use the numbers of simulations: $N_m = N_t = 5,000$. See the algorithms in Appendix A for more detail.
the ZLB binds.

According to the total variance method, the relative variance contributions of technology shocks to GDP and inflation decline substantially compared to the case of steady state, especially at very short horizons when the nominal interest rate hits the ZLB. For example, based on the total variance method, the 1-period horizon contribution of the technology shock to GDP declines to 2.25% in the Great Recession case, from 12.62% in the steady state case. The 1-period-horizon contribution of the technology shock to inflation also declines to 23.26% from 27.43%.

At longer horizons when the ZLB is not likely binding, the variance contribution of the technology shock to GDP increases, closer to that in the case of steady state. The contribution of technology shock to the variance of GDP at a 20-period horizon is around 7.00%, based on the total variance method.

To see why the variance contributions of technology shocks to GDP and inflation decline in the Great Recession, we compute and plot the generalized impulse response function (GIRF) under a one-standard-deviation shock to technology and preference (0.57 for technology and 0.25 for preference). Figure 3 shows the GIRFs for two cases: steady state and the Great Recession.

The left column of figure 3 shows the GIRFs for the nominal interest rate, GDP, and the inflation rate at the steady state, while the right column presents the GIRFs at the Great Recession. As explained above, at this Great Recession state, output gap is about −2.7% quarterly, the inflation rate is about −0.5% per year, and the ZLB binds.

It is notable to see that the responses of economic variables change substantially from the steady state to the Great Recession state. In particular, the response of GDP to a positive technology shock becomes smaller, even negative in this case. This result is consistent with Eggertsson and Krugman (2012), where they find that a rise in productivity
Figure 3: Generalized impulse response functions (GIRFs) under a positive one-standard-deviation technology shock and a positive one-standard-deviation preference shock. The GIRFs are computed based on 5,000 runs; each has 20 periods.
is contractionary at the ZLB due to the Fisher effect. Note that the response of GDP could still be positive if the magnitude of the technology shock was smaller than its one standard deviation.

More importantly, the response of GDP to a positive technology shock becomes much smaller relative to the response of GDP to a preference shock at the Great Recession state, in terms of absolute value. This explains why the contribution of the technology shock to GDP variance declines substantially at the ZLB compared to at normal times.

In comparing the two variance decomposition methods at the Great Recession state, from Table 2 we can see that compared to the total variance method, the GFEVD method produces very different results, especially at longer horizons. In particular, based on the GFEVD method, the relative importance of technology shock to GDP does not increase at longer horizons. For example, using the GFEVD method, the contribution of the technology shock to the variance of GDP at a 20-period horizon are still relatively small, around 1.69% compared to 7.00% from the total variance method. This means that the indirect contribution due to shock interactions plays an important role in quantifying relative variance contributions of shocks, especially at long horizons.

In general, the results from the GFEVD method are relatively off those generated by the total variance method, especially for GDP at longer horizons. The difference comes from the indirect variance contribution by each shock due to higher order effects, i.e. shock and state interactions.

4 Conclusion

This paper proposes a "new" method called the total variance method, together with algorithms to compute and analyze variance decomposition for nonlinear DSGE models. In addition, we compare our method with the only existing method, which is called the
GFEVD method proposed by Pesaran and Shin (1998) and then modified by Lanne and Nyberg (2016). For an empirical illustration, we apply these methods to a standard DNK model with an occasionally binding ZLB. More specifically, we study the relative importance of supply and demand shocks to business cycles in normal times and in the Great Recession when the ZLB binds.

Using our variance decomposition method, we find that the supply shock becomes significantly less important relative to the demand shock in explaining the volatility of economic variables, especially GDP, at a short horizon when the economy stays in a deep recession with binding ZLB. This occurs because the response of GDP to technology shock becomes much smaller relative to the response of GDP to preference shock at the Great Recession state with a binding ZLB.

More importantly, our theoretical and empirical comparison of the methods show that when indirect contributions of shocks are important due to shock multiplication and interactions, relative variance contributions of structural shocks vary substantially across the methods. Therefore, we recommend that when working with nonlinear DSGE models researchers should use the total variance method in order to see the importance of indirect variance contributions and to quantify correctly the relative variance contribution for each structural shock.

References


Appendix

A Algorithms for variance decompositions

In this section, we provide our algorithms for the two methods previously described. They are part of our contribution to the literature on nonlinear DSGE models.

Algorithm 1 The law of total variance method

1. Compute the variance contribution of shock \( j = 1, \ldots, k \), where \( k \) is the number of structural shocks.

(a) For \( m = 1, \ldots, N_m \), where \( N_m \) is the number of simulations for all the orthogonal shocks excluding shock \( j \) from \( t+1 \) to \( t+h \), \( \epsilon_{t+1:t+h}^{-j} \).

i. Simulate a path of all the orthogonal shocks excluding shock \( j \), \( \epsilon_{t+1:t+h}^{-j} \).

ii. For \( l = 1, \ldots, N_l \), where \( N_l \) is the number of simulations for shock \( j \) from \( t+1 \) to \( t+h \), \( \epsilon_{t+1:t+h}^{j} \).

- Simulate a path of shock \( j \), \( \epsilon_{t+1:t+h}^{j(l)} \), and compute \( Y_{t+h} \left( S_t, \epsilon_{t+1:t+h}^{(l,m)} \right) \) where \( \epsilon_{t+1:t+h}^{(l,m)} \) is the path of all the shocks combined; shock \( j \) from iteration \( l \) and the other shocks from iteration \( m \).

iii. Calculate the empirical expectation and variance of \( Y_{t+h} \left( S_t, \epsilon_{t+1:t+h}^{(l,m)} \right) \) contributed by shock \( j \) conditional on \( S_t \) and \( \epsilon_{t+1:t+h}^{-j,(m)} \):

\[
\hat{E}_t \left[ Y_{t+h} | \epsilon_{t+1:t+h}^{-j,(m)} \right] = \frac{1}{N_l} \sum_{l=1}^{N_l} Y_{t+h} \left( S_t, \epsilon_{t+1:t+h}^{(l,m)} \right)
\]

\[
\hat{V}_t \left[ Y_{t+h} | \epsilon_{t+1:t+h}^{-j,(m)} \right] = \frac{1}{N_l} \sum_{l=1}^{N_l} \left( Y_{t+h} \left( S_t, \epsilon_{t+1:t+h}^{(l,m)} \right) - \hat{E}_t \left[ Y_{t+h} | \epsilon_{t+1:t+h}^{-j,(m)} \right] \right)^2.
\]
(b) Compute the empirical variance caused by shock $j$ as

$$
\hat{\sigma}^2_{Y_t|Y_{t+h}} = \frac{1}{N_m} \sum_{m=1}^{N_m} \hat{\sigma}^2_t[Y_{t+h}|e^{-j(m)}_{t+1:t+h}],
$$

(c) Compute the empirical variance caused by all the shocks excluding shock $j$, the empirical counterpart of the second term in the RHS of (6):

$$
\hat{\sigma}^2_{Y_t|Y_{t+h}} = \frac{1}{N_m} \sum_{m=1}^{N_m} \left( \hat{E}_t[Y_{t+h}|e^{-j(m)}_{t+1:t+h}] - \hat{E}_t[Y_{t+h}] \right)^2,
$$

where

$$
\hat{E}_t[Y_{t+h}] = \frac{1}{N_m} \sum_{k=1}^{N_m} \hat{E}_t[Y_{t+h}|e^{-j(m)}_{t+1:t+h}],
$$

(d) Compute the empirical total variance of $Y_{t+h}$ conditional on state $S_t$:

$$
\hat{\sigma}^2_{Y_t|Y_{t+h}} = \hat{\sigma}^2_{Y_t|Y_{t+h}} + \hat{\sigma}^2_{Y_t|Y_{t+h}}.
$$

(e) Compute the empirical FEVD for shock $j$ at horizon $h$ conditional on state $S_t$ as

$$
\hat{\lambda}_t^j(h) = \frac{\hat{\sigma}^2_{Y_t|Y_{t+h}}}{\hat{\sigma}^2_{Y_t|Y_{t+h}}}. 
$$

2. Compute the empirical normalized FEVD for shock $j = 1, \ldots, k$:

$$
\tilde{\lambda}_t^j(h) = \frac{\hat{\lambda}_t^j(h)}{\sum_{i=1}^{k} \hat{\lambda}_t^i(h)},
$$

36
where $k$ is the number of orthogonal shocks.

**Algorithm 2** The GFEVD method

1. For $m = 1, ..., N_m$, where $N_m$ is the number of simulations of $\epsilon_{t+1}$ which is the vector of structural shocks at time $t+1$.

   (a) Simulate $\epsilon_{t+1}$, called $\delta^{(m)} = (\delta_1^{(m)}, ..., \delta_k^{(m)})$, where $k$ is the number of structural shocks.

   (b) For $l = 1, ..., N_l$, where $N_l$ is the number of simulations of $\epsilon_{t+1:t+h}$ which is the vector of orthogonal shocks from time $t+1$ to $t+h$.

   • Simulate a path of all the shocks, $\epsilon^{(l)}_{t+1:t+h}$.

   (c) For $j = 1, ..., k$, where $k$ is the number of structural shocks.

      i. Compute average responses

      $$\hat{E}_t [Y_{t+h} | \epsilon_{t+1}^j = \delta_j^{(m)}] = \frac{1}{N_l} \sum_{l=1}^{N_l} Y_{t+h} \left( S_t, \epsilon_{t+1}^j = \delta_j, \epsilon_{t+1}^{-j, (l)}, \epsilon_{t+2:t+h}^{(l)} \right),$$

      $$\hat{E}_t [Y_{t+h}] = \frac{1}{N_l} \sum_{l=1}^{N_l} Y_{t+h} \left( S_t, \epsilon_{t+1}^{(l)} \right).$$

      ii. Then compute generalized impulse responses for variable $Y_{t+h}$ under a shock to $\epsilon_{t+1}^j$, conditional on state $S_t$ and shock size $\delta_j^{(m)}$:

      $$\hat{G}I_t^j \left( h | \epsilon_{t+1}^j = \delta_j^{(m)} \right) = \hat{E}_t \left[ Y_{t+h} | \epsilon_{t+1}^j = \delta_j^{(m)} \right] - \hat{E}_t [Y_{t+h}],$$

   (d) Compute the empirical FEVD for shock $j$ to $Y_{t+h}$ conditional on state $S_t$ and shock
size $\delta_j^{(m)}$:

$$
\hat{\lambda}_t^{j}(h|\epsilon_{t+1}^j = \delta_j^{(m)}) = \frac{\sum_{i=1}^{h} \left( \hat{G}_t^{j} \left( l|\epsilon_{t+1}^j = \delta_j^{(m)} \right) \right)}{\sum_{i=1}^{k} \sum_{l=1}^{h} \left( \hat{G}_t^{i} \left( l|\epsilon_{t+1}^i = \delta_i^{(m)} \right) \right)^2}.
$$

(A.1)

2. Compute the FEVD for shock $j$ to $Y_{t+h}$ conditional on state $S_t$:

$$
\hat{\lambda}_t^{j}(h) = \frac{1}{N^m} \sum_{m=1}^{N^m} \hat{\lambda}_t^{j} \left( h|\epsilon_{t+1}^j = \delta_j^{(m)} \right).
$$

B Data description

We use the following data to filter the unobserved state (preference and technology):

- U.S. Real GDP: Chained 2012 dollars, seasonally adjusted. Source: Bureau of Economic Analysis, National Income and Product Accounts, Table 1.1.6. (FRED ID: GDPC1).

- U.S. GDP Deflator: Index 2012=100, seasonally adjusted. Source: Bureau of Economic Analysis, National Income and Product Accounts, Table 1.1.9. (FRED ID: GDPDEF).


C Solution method

Our solution method is close to, but slightly different from, the one used in Fernandez-Villaverde et al. (2015). Similar to their method, we do not approximate the policy
function for the nominal interest rate. Instead, the nominal interest rate is always deter-
determined by equation (30) at every state, in or out of the set of collocation nodes. However,
different from them, we approximate the expectations as a function of state using the
cubic spline interpolation; see Judd (1998) and Miranda and Fackler (2002) for more
details. The main advantage of this approach is that we do not have to worry about the
kink when the ZLB starts binding.

Following Miranda and Fackler (2002), we rewrite the functional equations governing
the equilibrium in a more compact form:

\[ f(s, X(s), E[Z(X(s'))]) = 0. \]  \hspace{1cm} (C.1)

where

- \( f : \mathbb{R}^{2+7+2} \rightarrow \mathbb{R}^7 \) is the equilibrium relationship;
- \( s = (\beta, A) \) is the current state of the economy;
- \( X(s) = (R(s), C(s), N(s), w(s), \Pi(s), Y(s), GDP(s))' \), and \( X : \mathbb{R}^2 \rightarrow \mathbb{R}^7 \) is the
  policy function, where \( R = 1 + i \) is the gross interest rate and \( \Pi = 1 + \pi \) is the
gross inflation rate.
- \( s' \) is the next period’s state that evolves according to the following motion equation:
  \[
  s' = g(s, \varepsilon) = \begin{bmatrix}
  \beta' = \beta^{\varepsilon_{\beta}} \exp(\varepsilon_{\beta}) \\
  A' = A^{\varepsilon_{\gamma}} \exp(\varepsilon_{A})
  \end{bmatrix},
  \]
  where \( \varepsilon_{\beta} \) and \( \varepsilon_{\gamma} \) are the innovations of the preference and the government spending
  shocks;
- \( Z(X(s')) = \begin{pmatrix}
  Z_1(X(s')) = \frac{C(s')^{-\gamma}}{\Pi(s')} \\
  Z_2(X(s')) = \frac{1}{C(s')^{-\gamma}} (\Pi(s') - 1) \Pi(s') Y(s')
  \end{pmatrix}. \]
Instead of solving policy function, we actually solve the expectations as a function of state using the cubic spline interpolation. Define $h(s) = E [Z(X(s'))|s]$, below is the simplified algorithm:

- **Step 1:** Define the space of the approximating functions and collocation nodes $S = (S_1, \ldots, S_N)$, where $N = N_{\beta} \times N_A$, and $N_{\beta}$ and $N_A$ are the numbers of grid points along each dimension of the state space. In this paper, we approximate the expectations:

$$ h(s) = (\phi(s)\theta_{h_1}, \phi(s)\theta_{h_2})' \text{ or } h(s) = \phi(s)\Theta, $$

where

- $\phi(s)$ is a $1 \times N$ matrix of cubic spline basis functions evaluated at state $s \in S = (S_1, \ldots, S_N)$.

- $\Theta = (\theta_{h_1}; \theta_{h_2})$ is a $N \times 2$ coefficient matrix that we want to approximate.

- **Step 2:** Initialize the coefficient matrix $\Theta^0$ and set up stopping rules.

- **Step 3:** At each iteration $j$ given the corresponding $\Theta^j$, we implement the following sub-steps:

  1. At each collocation node $s_i, s_i \in \{S_1, \ldots, S_N\}$, compute $h(s_i)$ using the approximating functions for the expectations.

  2. Solve for $X(s_i)$ such that $f(s_i, X(s_i), h(s_i)) = 0$. We solve this complementarity problem using the Newton method.

- **Step 4:** Update $h$ using the following sub-steps:
1. Approximate policy functions for $C, \Pi, Y$ using a cubic spline interpolation.

2. At each collocation node $s_i$, $s_i \in \{S_1..S_N\}$, update $h(s_i) = (h_1(s_i), h_2(s_i))$ using

\[
h_1(s_i) = \sum_{j}^{25} w_j \left[ \frac{C(s')^{-\gamma}}{\Pi(s')} \right]
\]

(C.2)

\[
h_2(s_i) = \sum_{j}^{25} w_j \left[ \frac{(\Pi(s') - 1) \Pi(s') Y(s')}{C'(s')^{-\gamma}} \right]
\]

(C.3)

where the innovations for the preference and government spending shocks are discretized using the Tauchen and Hussey (1991) method with 25 nodes.

- **Step 5:** Update $\Theta^{j+1} = \Phi^{-1}\Theta^j$, where $\Phi = (\phi(s_1), ..., \phi(s_N))^\prime$.

- **Step 6:** Check the stopping rules. If not satisfied go to Step 3; otherwise go to Step 7.

- **Step 7:** Report results. We use the approximated expectation functions to solve for the equilibrium value at any state. So, we are able to find almost exactly the kink for the nominal interest rate.

In addition, we write our code using a parallel computing method that allows us to split up a large number of collocation nodes into smaller groups assigned to different processors to be solved simultaneously. This procedure reduces computation time significantly. We obtain the maximal absolute residual across the equilibrium conditions of the order of $10^{-8}$ for almost all states off the collocation nodes. For a few states when the ZLB becomes binding, the maximal absolute residual is of the order of $10^{-5}$. This is quite standard given the kink in the interest rate policy function; see Miranda and Fackler (2002) and Judd et al. (2011) for more information.