
1. A discrete π may be looked at algebraically and geo-topologically.

2. Algebra.
   Let \( R \text{ ring}, \ M \text{ } R\text{-module} \)
   \( \text{Proj dim}_R M \leq n \) if there is a projective resolution of \( M \) of length \( n \):
   \[
   0 \to P_0 \to P_{n-1} \to \cdots \to P_i \to P_0 \to M \to 0
   \]
   \( H_i(P) = 0 \) \( i \geq 1 \), \( H_0(P) = \frac{P_{0}}{\ker(P \to P_0)} \cong M \).
For \( \pi \) a discrete group, we take the ring to be the group ring \( \mathbb{Z}[\pi] \). Take \( \mathbb{Z} \) to be a trivial \( \mathbb{Z}[\pi] \)-module.

The cohomological dimension of \( \pi \) is

\[
\text{cd} (\pi) = \text{projdim}_{\mathbb{Z}[\pi]} \mathbb{Z} = \inf \{ n : \mathbb{Z} \text{ admits a resolution by } \mathbb{Z}[\pi] \text{-modules of length } n \}.
\]

Recall how we find cohomology of \( \pi \):

1) Take a resolution of \( \mathbb{Z} \),

\[
\mathbb{Z} \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z}
\]

2) Take a \( \mathbb{Z}[\pi] \)-module \( A \)
3) Form the complex
\[ \text{Hom}_{\mathbb{Z}^+}(\mathbb{Z}^+, A) \rightarrow \text{Hom}_{\mathbb{Z}^+}(\mathbb{P}_0 A) \rightarrow \ldots \rightarrow \text{Hom}_{\mathbb{Z}^+}(\mathbb{P}_j A) \rightarrow \ldots \]

4) \[ H^i(\pi, A) \overset{\text{def}}{=} H_i(\text{Hom}_{\mathbb{Z}^+}(\mathbb{P}_j A)) \]

(Exercise: \( H^0(\pi, A) = A^\pi = \text{fixed submodule} \))

\[ \Rightarrow \text{cd}(\pi) = \inf \{ n : H^i(\pi, -) = 0 \text{ for } i > n \} \]
\[ = \sup \{ n : H^i(\pi, M) \neq 0 \text{ for some } M \} \]

3. Topological
If \( \pi \) has a presentation \( 0 \rightarrow R^jF \rightarrow \pi \)
with \( R, F \) free, then we can model
This by Ven-Kampeien's Theorem says
\[ \pi_1(X_2) = \pi. \]

It is possible that \( \pi_2(X_2) \neq 0 \). If so, attach 3-cells to kill this group. We get \( X_3 \) with \( \pi_1(X_3) = \pi \), \( \pi_2(X_3) = 0 \).

Continue to attach cells until we have \( X \) with \( \pi_1(X) = \pi \) and \( \pi_j(X) = 0 \) \( \forall j > 1 \).

Then \( X \) is called a \( K(\pi, 1) \).
The geometric dimension of $\pi$ is the smallest $n$ such that there is a $K(\pi, 1)$ of dimension $n$. This is written $\dim(\pi)$.

Example, $\pi = \mathbb{Z} \times \mathbb{Z}$, $F_1 \rightarrow F_2 \rightarrow \mathbb{Z} \times \mathbb{Z}$

Start with $S^1 \vee S^1$.

$S^1 \rightarrow S^1 \vee S^1 \rightarrow X_2 = S^1 \times S^1 = T^2$

$\omega \mapsto aha^{-1}b^{-1}$

$T^2$ is covered by $\mathbb{R}^2 \Rightarrow T^2 = K(\mathbb{Z} \times \mathbb{Z}, 1)$.
4. What is a connection between the algebraic and geometric approaches?

Look at $\tilde{X} \to X = \text{ker}(\pi_j)$. Since $\pi_j(\tilde{X}) = \pi_j(X) = 0$ for $j > 1$ and $\pi_0(\tilde{X}) = 0$, we see $\tilde{X}$ is contractible.

Consider the chain complex of cells of $X$:

$\cdots \to C_k(X) \to C_{k-1}(X) \to \cdots \to C_1(\tilde{X}) \to C_0(\tilde{X}) \to \mathbb{Z} \to 0$

$C_i(\tilde{X}) = |\pi_i|$ - number of cells above each $i$-cell in $X$.

This gives a resolution of $\mathbb{Z}$ by free
$\mathbb{Z}/\pi$-modules.

Then $H^i(\pi, A)$ is then computed by

$$\cdots \to \text{Hom}_{\mathbb{Z}/\pi}(\mathbb{C}_i(X), A) \to H_{\mathbb{Z}/\pi}^i(\mathbb{C}_i(X), A) \to \cdots$$

If $A = \mathbb{Z}/\pi$, then $H^*_c(X) = H^*_i(\pi, \mathbb{Z}/\pi)$.

If $A = \mathbb{Z}$ (trivial), then $H^*_c(\pi, \mathbb{Z}) = H^*_c(X, \mathbb{Z})$.

$\text{alg def of } \text{coh of } \varpi$$

$\text{ordinary sing.}$

Suppose $\dim_{\mathbb{C}}(X) = n$,

$\Rightarrow \mathbb{C}_i(X) = 0$ for $i > n$,

$\Rightarrow H^i_c(\pi, A) = 0$ for $i > n$.

$\Rightarrow \text{codim}(\pi) \leq \dim_{\mathbb{C}}(X)$.
Theorem: (Eilenberg-Ganea)
Let \( \pi \) be an arbitrary group and let \( n = \max \{ \text{cd}(\pi), 3 \} \). Then there exists an \( n \)-dim \'K(\pi_1)\)-complex \( Y \).
If \( \pi \) is finitely presented and of type \( \text{FL} \) (resp. \( \text{FP} \)), then \( Y \) may be taken to be finite (resp. finitely dominated).

Corollary. If \( \text{cd}(\pi) \geq 3 \), then \( \text{cd}(\pi) = \dim(\pi) \).
EG - Conjecture. \( \text{cd}(\pi) = \dim(\pi) \) always.
Could there be a \( \pi \) with \( \text{cd}(\pi) = 2 \), \( \dim(\pi) = 3 \)?
Lusternik-Schnirelmann category enters this picture.
By the fibre-cofibre formulation of LS category, we can show

**Theorem.** If $X = \text{K}(\pi, 1)$, then

$$\text{cd}(\pi) = \text{cat}(X).$$