Finite Element Exterior Calculus

Finite element method $\rightarrow$ “solving” pde’s.

Here is an example from topology - geometry.

Ex. Let $M$ be a compact Riemannian manifold. Is there a closed geodesic on $M$?

Let $\Lambda M = \mathbb{M}^5$ be the free loop space. There is a Hilbert manifold version of this and on this “manifold” there is an
energy functional \( E(\alpha) = \int_0^T \| \dot{\alpha} \|^2 \, dt \).

The extremals of \( E \) (i.e. critical points) are the closed geodesics. In particular, they are the constant loops — these correspond to a section \( \mathcal{M} \rightarrow \Lambda \mathcal{M} \) of the fibration \( \Lambda \mathcal{M} \rightarrow \mathcal{M} \).

This is a hard problem — analytically.

Here is the Morse-Bott approach.

\[
P_n(M) = \left\{ (x_1, \ldots, x_n) \in \mathcal{M}^n \mid d(x_1, x_2) + \ldots + d(x_1, x_n)^2 \leq \varepsilon^2 \right\}
\]

where \( \varepsilon = \text{injectivity radius of } \mathcal{M} \).
This gives a geodesic polygon. Note that $P_n(M)$ is compact (so finite-dimensional).

Define $E' : P_n(M) \to \mathbb{R}$ by

$$E'(x_1, \ldots, x_n) = \sum_{i=1}^{n} d(x_i, x_{i+1})^2 \quad x_{n+1} = x_1.$$  

$E'$ discretizes $E$.

(1) $P_n(M)$ is compact.

(2) A critical point of $E'$ gives a polygon with no corners — a closed geodesic.

(3) We can take $n$ large enough so that

$$\bar{\pi}_k(P_n(M)) \xrightarrow{\text{h}} \bar{\pi}_k(M) \quad \text{for } k \leq \frac{N}{n}. \quad (\text{Bott})$$
Theorem (Lusternik– Fet)
On any compact simply connected Riemannian manifold, there exists a closed geodesic.

Proof. Suppose $E$ has no nontrivial critical points. This is a Morse theory situation.
No crit pts (and compactness)

$\Rightarrow$ Can deform $P_n(M)$ into $M$

$P_n(M) \cong M$.

Let $r$ be the smallest degree with $\pi_r(M) \neq 0$.

We know $\pi_i(\Lambda M) = \pi_i(\Sigma M) \oplus \pi_i(M)$

$= \pi_{i+1}(M) \oplus \pi_i(M)$. 
Choose n large enough so that
\[ \Pi_j (P_n(M)) \cong \Pi_j (AM) \quad 1 \leq h \leq j \leq N. \]
\[ \Rightarrow 0 \neq \Pi_r (M) \cong \Pi_{r-1} (AM) \cong \Pi_{r-1} (P_n(M)) \cong \Pi_{r-1} (M) \]
Contradiction,
\[ \Rightarrow E' \text{ has nontrivial points.} \]

Now let's talk about the FEM. Let's look at an example (which will give us a paradigm for the approach).
Consider the BVP: \[ \begin{cases} -u'' + u &= f \quad \text{on } (0,1), \\ u(0) = u(1) &= 0, \quad f \text{ given}. \end{cases} \]

Def. A \underline{classical solution} to \# is a function \( u \) satisfying \#. 

A \underline{weak solution} of \# is a function \( u_0 \) satisfying 
\[
\int_0^1 u_n' \, u_0' \, dt = \int_0^1 f_n \, u_0 \, dt
\] 

for all \( N \) with \( N(0) = 0 = N(1) \).

(1) A classical solution is a weak solution. 
(Integration by parts.)
(2) A weak solution is a classical solution.

\[ \int_t^1 u' n' + u n \, dt = \int_0^1 [-u'' + u] n \, dt \]

\[ \int_0^1 f n \, dt \quad \Rightarrow \quad \int_0^1 [-u'' + u - f] n \, dt = 0 \]

\[ \Rightarrow -u'' + u = f \quad \forall \, n \quad \text{with} \quad n(0) = 0 = n(1) \]

(3) A weak solution exists and is unique and is obtained by

\[ \min_n \int = \frac{1}{2} \int_0^1 u'^2 + u^2 - fu \, dt \]

This is called Dirichlet's Principle.
So what does a pde'er do? Use an approximation method — called Galerkin's Method.

Suppose \( \{N_0, N_1, N_2, \ldots \} \) (perhaps a finite #) are a basis of functions satisfying the boundary conditions

Let \( u = N_0 + \sum c_i N_i \) be a trial function. Plug into the weak solution integral with one of the \( N_j \)'s as \( N \).
\[
S'(\Sigma c_i \eta_i)' \eta_j' + (\Sigma c_i \eta_i) \eta_j \, dt = \int_0^b f \eta_j \, dt,
\]
\[
\Sigma c_i \left( S' \eta_i \eta_j' + \eta_i \eta_j \right) \, dt = \int_0^b f \eta_j \, dt
\]
\[
a_{ji}
\]
\[
b_j
\]
\[
\Rightarrow \Sigma a_{ji} c_i = b_j
\]
\[
\Rightarrow \mathbf{A} \mathbf{c} = \mathbf{b}, \quad \text{Solve for } \mathbf{c} = (c_1, …, c_n)
\]
\[
\text{Then } \mathbf{u} = \Sigma c_i \eta_i.
\]
This approximates a true solution.
Next week we will see where the weak solution integral comes from.