Lecture 2. Finite Element Ext. Calc. II

Review of Calculus of Variations.

Basic Problem. Minimize \( J = \int_{x_0}^{x_1} f(x, y(x), y'(x)) \, dx \)

Suppose \( y(x) \) is a minimizer for \( J \) and vary it by letting \( \bar{y}(x) = y(x) + \varepsilon \eta(x) \) where \( \eta(x_0) = 0 = \eta(x_1) \).

\[
\Rightarrow J[\varepsilon \eta] = \int_{x_0}^{x_1} f(x, \bar{y}, \bar{y}') \, dx \\
\frac{d J[\varepsilon \eta]}{d \varepsilon} = \int_{x_0}^{x_1} \frac{df}{d\bar{y}} \frac{d\bar{y}}{d\varepsilon} + \frac{df}{d\bar{y}'} \frac{d\bar{y}'}{d\varepsilon} \, dx
\]
\[ 0 = \frac{d J[\phi]}{d \varepsilon} \bigg|_{\varepsilon = 0} = \int_0^x \frac{df}{dy} \psi + \frac{df}{dy}, \psi \bigg|_0^x \, dx \]

This is the weak solution equation.

Aside: Suppose you have an inner product on an \( \infty \)-dimensional function space \( V \), \((\cdot, \cdot)\).

A solution \( \Delta w = f \) satisfies:

\[
(\Delta w, \phi) = (f, \phi)
\]

\[
(w, \Delta^* \phi) = (f, \phi)
\]

\[
(\Delta^* \phi) = (f, \phi)
\]

Fix \( w \) and define a functional \( I : V \to \mathbb{R} \):

\[
I(\alpha) = (w, \alpha)
\]

This is a weak solution for \( \Delta w = f \) in the sense of functional analysis.
We integrate \( S \frac{df}{dy} n \, dx \) by parts.

\[
S \frac{df}{dy} n = \int_{x_0}^{x_1} \frac{df}{dy} n - S \int_{x_0}^{x_1} \frac{d}{dx} \left( \frac{df}{dy} \right) n \, dx
\]

\[
= 0 - S_{x_0}^{x_1}
\]

Plug in above:

\[
0 = \int_{x_0}^{x_1} \frac{df}{dy} n - \frac{d}{dx} \left( \frac{df}{dy} \right) n \, dx
\]

\[
0 = \int_{x_0}^{x_1} n \left[ \frac{df}{dy} - \frac{d}{dx} \left( \frac{df}{dy} \right) \right] \, dx
\]

This is true for every \( N \) with \( N(x_0) = 0 = N(x_1) \),

\[
\Rightarrow \frac{df}{dy} - \frac{d}{dx} \left( \frac{df}{dy} \right) = 0
\]

Euler–Lagrange Equation.
Ex. \(-u'' + u = f\)

is the EL-eq for \(J = \int \frac{1}{2}(u')^2 + \frac{1}{2}u^2 - fu \, dx\)

EL. \(\frac{df}{du} - \frac{d}{dx}(\frac{df}{du'}) = u - f - \frac{d}{dx}(u') = 0\)

\(f = \text{integrand}\)

\(-u'' + u = f\) \(\checkmark\)

Weak sol. \(0 = \int \frac{df}{du}n + \frac{df}{dx}u' \, dx\)

\(0 = \int (u - f)n + uu' \, dx\)

\(\int fn \, dx = \int u'n' + un \, dx\) \(\checkmark\)

Ex. \(\Delta u(x,y) = f\) \(u = 0\) on unit square.

\(J = \iint \frac{1}{2}(u_x)^2 + \frac{1}{2}(u_y)^2 + fu \, dx \, dy\).
A Hodge Theory Primer

Let $M$ be a compact Riemannian manifold with metric $g$.

Let $T(M) = \bigcup_{x \in M} T_x(M)$, where $T_x(M)$ is the tangent space of $M$ at $x$.

Let $T_x(M)^*$ be the dual of $T_x(M)$ and form the exterior algebra $\bigwedge^* T_x(M)^*$.

Let $\Omega^*(M) = \bigcup_{x \in M} \bigwedge^* T_x(M)^*$.

These are the differential forms on $M$.

Ex. A 1-form $\phi$ operates on vector fields.

For $V$ a vector field, $\phi(V)(x) = \phi_x(V_x) \in \mathbb{R}$.

Ex. $x_i : M \to \mathbb{R}$, $dx^i = 1$-form.
In fact there is an exterior derivation
\[ d_j : \Omega^j \rightarrow \Omega^{j+1} \] 
with \( d_{j+1} \circ d_j = 0 \) \( \forall j \).

This means we have a complex
\[ 0 \rightarrow \Omega^0 \overset{d}{\rightarrow} \Omega^1 \overset{d}{\rightarrow} \Omega^2 \overset{d}{\rightarrow} \cdots \rightarrow \Omega^n \rightarrow 0. \]

The \textbf{de Rham cohomology} is given by
\[ H^j_{\text{DR}}(M) = \frac{\ker (d_j : \Omega^j \rightarrow \Omega^{j+1})}{\text{im} (d_{j+1} : \Omega^{j+1} \rightarrow \Omega^j)} \]
for \( j = 0, \ldots, n \).
Also, $\Omega^*(M)$ is a graded (commutative) algebra. If $\omega \in \Omega^k(M)$, $\phi \in \Omega^l(M)$, then $\omega \wedge \phi \in \Omega^{k+l}(M)$. Furthermore, $\omega \wedge \phi = (-1)^{kl} \phi \wedge \omega$.

Also $d(\omega \wedge \phi) = d\omega \wedge \phi + (-1)^k \omega \wedge d\phi$.

$\Rightarrow \mathcal{H}^{*}_{\text{DR}}(M)$ is a graded commutative algebra.

Stokes's Theorem

If $M^n$ has boundary $\partial M$, then for $\alpha \in \Omega^{n-1}(M)$, $\int_M \alpha = \int_{\partial M} d\alpha$. 
Integration gives a map from $\Omega^p(M)$ to $\mathcal{C}^p(M; \mathbb{R}) = p$-cochains with coeff. in $\mathbb{R}$ dual vector space to $C_p(M)$.

$S : \Omega^p(M) \rightarrow \mathcal{C}^p(M; \mathbb{R})$

$S(\omega(\sigma_p)) = S\omega$

$p$-cell

Note: $S \omega(\sigma_p) = Sd\omega = S\frac{\partial \omega}{\partial \tau_p}$

$\Rightarrow$ $S$ is a chain map.
\[ \Rightarrow S^*: H^*_\text{DR}(M) \to H^*(M; \mathbb{R}) \]

**Theorem (de Rham)**

*\(M\) smooth without boundary

\[\Rightarrow \quad lH^*_\text{DR}(M) \xrightarrow{\cong} S^* \xrightarrow{\cdot} H^*(M; \mathbb{R}) \text{ as algs.} \]