Lecture 4: Cochain Finite Elements, the Whitney Map and RHT.

Let \( X \) be a smooth Riemannian manifold that is \( C^\infty \)-triangulated by a simplicial complex \( K \). There is a fixed ordering on the vertices of \( K \), \( X \) compact \( \iff K \) finite. \( K \) finite \( \Rightarrow \) we can identify cochains and chains. Let \( c \in C^g(K) \). Write

\[
    c = \sum c_\tau \tau \quad \text{where} \quad \tau = g\text{-simplex of } K,
\]

Let \( \tau = [p_0, \ldots, p_g] \), \( p_i \) vertices of \( K \).
Define a map $W: C^\infty(K) \to \Omega^\infty(X)$ by

$$W\gamma = \sum_{k=0}^{\infty} \gamma^{(k)} \mu_{p_0} \cdots \mu_{p_k} \wedge \cdots \wedge \mu_{p_0}.$$ 

where the $\mu_{p_i}$ are the barycentric coordinates of $\gamma$.

This means any point $x \in X$ may be written as $x = \mu_{p_0} \cdot p_0 + \cdots + \mu_{p_k} \cdot p_k$ when $\mu_{p_i} \geq 0$, $\sum \mu_{p_i} = 1$.

Really $W: C^\infty(K) \to L^2 \Omega^\infty(X)$ completion of $\Omega^\infty(X)$ with respect to the inner product.
$W$ is the Whitney map.

Let integration be denoted by $R: \Omega^b(X) \to C^b(k)$, $R(\phi)(\gamma) = \int \phi$. 

Here are some properties of $W$.

(1) $W\gamma = 0$ on $X \setminus St(\gamma)$

(2) $Wd\alpha = d(W\alpha) \forall$ cochains $\alpha$

(3) If $\sigma, \tau$ are $q$-simplices with $p = \sigma \cup \tau$, $i: \epsilon \to \sigma$, $j: \epsilon \to \tau$, then...
\[ i^*(Wc|\sigma) = j^*(Wc|\mu) \]

**Proof**

\[ i^* \mu = \mu|_{\mathcal{L}} = j^* \mu \]

in barycentric coords and d commutes w. restriction.

(4) Let \( c \in C^g(K) \), \( a \in C^f_p(K) \). Then

\[ R(Wc)(a) = \sum_{a} Wc = \langle c, a \rangle = \text{eval of a cochain on a chain.} \]

(5) \( RW = id \).

(6) \( (dWc, f) = (Wc, df) \), \( f \in \Omega^{g+1}(X), c \in C^g(K) \).
The triangulation has a mesh
\[ \mathcal{N}_n = \sup_{\sigma \in S_n^K} \text{diam}(\sigma) \]
where \( S_n^K \) is a certain standard subdivision of \( K \).
\[ \lim_{n \to \infty} \mathcal{N}_n = 0 \]
Take \( n \) big for a good approx.

**Theorem.** Let \( f \in L^\infty(K) \). Then \( f \) has a constant \( C_f \) independent of \( n \) such that
\[ \| f - W_n f \|_1 \leq C_f \mathcal{N}_n. \]

Let's define an inner product on \( C^\infty(K) \) by
\((c, c') \overset{\text{def}}{=} (Wc, Wc') = S \times Wc \perp *Wc'\)

Now everything we did on Hodge theory applies to \((C^g(K), (,))\).

\[ C^g(K) = \bigoplus_{S, K} \mathcal{H}_n \oplus \text{Im}(d_n) \oplus \text{Im}(\partial_n) \]

\(n\) refers to \(S^n K\).

**Theorem (Dodziuk)**

Let \(f \in \Omega^g(X)\) have Hodge decomposition

\[ f = h + dg + \partial K. \]

Then \(R^*_n f\) has discrete Hodge decomposition

\[ R^*_n f = h_n + d_n g_n + \partial_n K_n. \]
Then
\[ \lim_{n \to \infty} W_n h_n = h, \quad \lim_{n \to \infty} W_n d_n g_n = dg, \quad \lim_{n \to \infty} W_n \varepsilon_n k_n = 2k, \]
in the norm of \( L^2(\Omega)(X) \).

Dodziuk mentions that his approach does not seem to apply to finding solutions for \( \Delta u = f \).
Discretely \( \Delta_n u_n = R_n f \), but we can't see that \( u_n \to u \).
This is the exact problem we want to solve!