MCE371: Vibrations

Prof. Richter

Department of Mechanical Engineering

Handout 7

Fall 2017

Cleveland State University

Free Undamped Vibration

Follow Palm, Sect. 3.2, 3.3 (pp 120-138), 3.5 (pp 144-151), 3.8 (pp. 167-169)

The equation of motion for unforced (free) and undamped 1-DOF systems is

$$m\ddot{x} + kx = 0$$

The general solution has the form $x = A\sin(wt + \phi)$

- Verify that the above solution fits the differential equation if $k = mw^2$
- The frequency

$$w = \sqrt{\frac{k}{m}}$$

  has a key role in vibrations. It's called the *natural frequency*

Rayleigh's Method

Sometimes, we are interested in finding the natural frequency without having to derive the equations of motion. This can be done using Rayleigh's method **if there is no friction** (energy-conserving vibrations). By conservation of total energy, the kinetic energy is minimum (zero) when the potential energy is maximum, and the kinetic energy is maximum when the potential energy is minimum. This means

$$T_{max} + V_{min} = 0 + V_{max}$$

so

$$T_{max} = V_{max} - V_{min}$$

In harmonic motion, the points of zero kinetic energy are reversal points, where motion amplitude is maximum. If the displacement amplitude is A and the frequency is w_n, the velocity amplitude will be Aw_n.

Using Rayleigh's Method

When spring force (and not gravity) is the restoring force causing oscillation, we can ignore gravity effects and consider oscillations about the equilibrium position. In these cases, the frequency of oscillation is independent of g.

- Define an origin of coordinates for x at the equilibrium position and assume $x = A\sin(wt)$.
- Find expressions for the kinetic energy and the *elastic* potential energy as a function of x and $\dot{x}$.
- Find T_{max}, V_{min} and V_{max}.
- Use $T_{max} = V_{max} - V_{min}$ to find the frequency (A should drop out of the equation)

Example

Mass m hanging from a spring k.

- Define the origin at the rest position and assume $x = A\sin(wt)$. Then $\dot{x} = Aw\cos(wt)$.
- $T = \frac{1}{2}m\dot{x}^2$ and $V = \frac{1}{2}kx^2$.
- $T_{max} = \frac{1}{2}m(Aw)^2$, $V_{min} = 0$, $V_{max} = \frac{1}{2}kA^2$
- Equation: $\frac{1}{2}m(Aw)^2 = \frac{1}{2}kA^2$. A^2 cancels out.
- We find $w^2 = k/m$, as expected. The frequency is the same on earth, the moon or outer space.

Examples: P3.31

Note: In P3.36, gravity is the *only* restoring force, can't ignore it!

Free 1-DOF Vibration with Viscous Damping

The equation of motion always has the form

$$m\ddot{x} + c\dot{x} + kx = 0$$

In Laplace form

$$(ms^2 + cs + k)X(s) = 0$$

- The form of the solution depends on the roots (poles) of the *characteristic equation* $ms^2 + cs + k = 0$
- From the quadratic formula:

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

- If $c^2 - 4mk > 0$ we get 2 distinct real poles (overdamped case)
- If $c^2 - 4mk = 0$ we get 2 repeated real poles (critically-damped case)
- If $c^2 - 4mk < 0$ we get a pair of complex-conjugate poles (underdamped case)

Overdamped Solution

$$x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

where $s = r_1$ and $s = r_2$ are the real and distinct poles.

- If at least one pole is positive, the solution grows to infinity (not regarded a "vibration")
- If both poles are negative, $x(t)$ dies out to zero as t increases.
- We can find A_1 and A_2 if the initial conditions $x(0)$ and $\dot{x}(0)$ are known, see Palm p.136.

Critically-Damped Solution

$$x(t) = A_1 e^{r_1 t} + t A_2 e^{r_1 t}$$

where $s = r_1$ is the real repeated pole (ch. eq. is $(s - r_1)^2 = 0$).

- If the pole is positive or zero, the solution grows to infinity.
- If the pole is negative, $x(t)$ dies out to zero as t increases.
- We can find A_1 and A_2 if the initial conditions $x(0)$ and $\dot{x}(0)$ are known, see Palm p.137.

Underdamped Solution

$$x(t) = Be^{rt}\sin(qt + \phi)$$

where $s = r \pm qi$ is the complex-conjugate pole pair (solution of $ms^2 + cs + k = 0$).

- If the real part of the pole (r) is positive or zero, the solution grows to infinity.
- If the pole is negative, $x(t)$ dies out to zero as t increases.
- We can find B and ϕ if the initial conditions $x(0)$ and $\dot{x}(0)$ are known, see Palm p.137.

The Standard 1-DOF Equation

Assuming viscous damping and free vibration, we've seen that the equation is

$$m\ddot{x} + c\dot{x} + kx = 0$$

Dividing through by m:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

Define the following constants

$$\zeta = \frac{c}{2\sqrt{mk}} \text{ the } \textit{damping ratio}$$

$$w_n = \sqrt{\frac{k}{m}} \text{ the } \textit{natural frequency}$$

With these definitions the standard equation of motion is

$$\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = 0$$

Solutions According to Damping Ratio

Comparisons of $c^2 - 4mk$ against zero are equivalent to comparisons of ζ against 1, as follows:

- If $\zeta > 1$, the solution is overdamped
- If $\zeta = 1$, the solution is critically-damped
- If $0 < \zeta < 1$, the solution is underdamped
- If $\zeta = 0$ there is no damping (pure harmonic motion).
- If $\zeta < 0$ the solution grows to infinity (unstable).

Underdamped Vibration: The Damped Natural Frequency

The poles in the underdamped case have the form

$$s = -\zeta w_n \pm \sqrt{1-\zeta^2} w_n i$$

The free response in the underdamped case is a decaying oscillation (recall 1st handout)

$$x(t) = Be^{-\zeta w_n t} \sin(w_d t + \phi)$$

where w_d is the *damped natural frequency:*

$$w_d = \sqrt{1-\zeta^2}$$

- w_d (not w_n) is the frequency that is actually observed in a decaying oscillation
- We can find w_d by measuring the period T_d of the oscillation and $w_d = \frac{2\pi}{T_d}$
- If w_d and ζ are known, we can find w_n easily.

Underdamped Vibration: Logarithmic Decrement vs. Damping Ratio

Logarithmic decrement: $\delta = \ln \frac{x(t)}{x(t+T)}$, T is the period.

For non-consecutive peaks: $\delta = \frac{1}{n} \ln \frac{B_1}{B_{n+1}}$ (see Palm, p.150)

Relationship to ζ:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

Coulomb (Dry) Friction

An mass sliding on a surface with Coulomb friction (μN type) is affected by a force of constant magnitude and changing direction. The direction of the friction force is always opposite to the direction of travel. We take the friction force to be μN, where N is the normal force (component of weight perpendicular to velocity vector). The friction force can be expressed as

$$F_f = -\mu N \text{ sign } (\dot{x})$$

The equation of motion is then

$$m\ddot{x} + kx = -\mu N \text{ sign } (\dot{x})$$

Free Vibration with Coulomb Friction Solutions

Assume the initial conditions are $x(0) = x_0 > 0$ and $\dot{x}(0) = 0$. If the restoring force can overcome friction ($kx_0 > \mu N$), the block moves with negative velocity and we have

$$m\ddot{x} + kx = \mu N$$

The solution has the form

$$x(t) = (x_0 - \mu N/k)\cos(w_n t) + \mu N/k$$

which is valid only while $\dot{x} \leq 0$ (until the block reaches zero velocity). The time for this to happen can be easily calculated: $t = \pi/w_n$. Then the block begins moving with positive velocity, and we have

$$m\ddot{x} + kx = -\mu N$$

The solution in this case is

$$x(t) = (x_0 - 3\mu N/k)\cos(w_n t) - \mu N/k$$

which is valid only for $\pi/w_n \leq t \leq 2\pi/w_n$.

Free Vibration with Coulomb Friction Solutions...

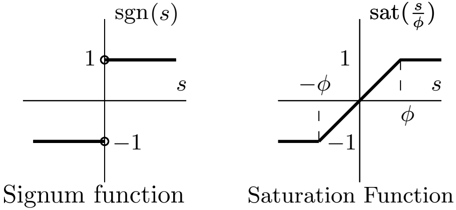
Continuing the solution, it is seen that the solution segments are cosine functions with decreasing amplitudes:

- The sequence of amplitudes is $X_{n+1} = X_n - 4\mu N/k$ (the envelope is a straight line). The initial amplitude is x_0.
- The observed frequency in this case **is** w_n.
- Motion will continue until x falls below $\mu N/k$. Then the mass gets stuck at this value.

Exercise: Sketch the shape of the oscillations in Prob. 3.70.

Simulating Vibrations with Coulomb Friction

The sign function represents a sharp discontinuity. It is likely to cause solvers like `ode45` to crash. We can use a continuous approximation to the sign function: the *saturation function*.



The idea is to divide $\dot{x}$ by a small constant ϕ and then apply the saturation function. The slope of the approximation is $1/\phi$. We can make ϕ as small as required to achieve a good tradeoff between numerical solution speed and accuracy.

Example

We simulate the equation of motion for Example 3.70 (take $m = 1$) in Simulink using the saturation function. Note the difficulty in obtaining an accurate stopping point.