Fenn College of Engineering
Mechanical Engineering Department
Cleveland State University

DRE QUALIFYING EXAMINATION

Spring, 2001

Mathematics Test (Open Book)

Name:

PART I: Differential Equations

Note: \( y = y(x), \ y' = \frac{dy}{dx}, \) and \( y'' = \frac{d^2y}{dx^2}. \) Find the solution.

1. \( xy' = \frac{y^2}{x}, \) for \( x \neq 0 \)
2. \( y' = 8x^3y^2 \)
3. \( 2xy' + y = 2 \)
4. \( y'' + 4y' + 4y = 0 \)
5. \( y'' + 2y' + 6y = 0 \)
6. \( y'' + 2y' - 3y = 4e^{2x} \)
7. \( x^2y'' + 2xy' - 6y = 0, \) for \( x > 0 \)
8. \( 4xy + 2x + (2x^2 + 3y^2)y' = 0 \)
9. \( xy' = x^3 + y \)
10. \( y'' + (y')^2 + 1 = 0 \)

PART II: Sequences

Find the limit.

1. \( \lim_{n \to \infty} \frac{2^n}{4^n + 1} \)
2. \( \lim_{n \to \infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) \)
3. \( \lim_{n \to \infty} \left( 1 + \frac{2^n}{n^2} \right) \)
4. \( \lim_{n \to \infty} \left( \frac{3^n}{4} \right) \)
PART III: Series
Find the sum.
1. \[ \sum_{k=1}^{\infty} \left( \frac{3}{4} \right)^k \]
2. \[ \sum_{k=1}^{\infty} \frac{e^k}{k!} \]
3. \[ \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{2} \right)^k \]

PART IV: Vector Calculus
1. Find a unit vector perpendicular to the plane that contains both \( \vec{A} = \vec{i} - 2\vec{j} \) and \( \vec{B} = \vec{i} - \vec{k} \).
2. Find a unit vector in the direction of the maximum rate of change of \( \phi = x^2 - 2yz + xy \) at the point (2,-1,-1).
3. Find the projection of \( \vec{A} \) in the direction of \( \vec{B} \) if \( \vec{A} = 14\vec{i} - 7\vec{j} \) and \( \vec{B} = 6\vec{i} + 3\vec{j} - 2\vec{k} \).
4. The equation of plane perpendicular to and passing through the end of the vector \( \vec{A} = 2\vec{i} - 4\vec{j} + 6\vec{k} \) is given by \( 2x - 4y + 6z = k \). Determine value of k.
5. Find a unit vector normal to the surface \( x^2 + 3y^2 - 3z = 4 \) at the point (1,1,0).
6. Determine the divergence and curl of \( \vec{u} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k} \) at the point (1,1,1).

PART V: Complex Variables
1. Given: \( z_1 = 3 + 4i \) and \( z_2 = 4 + 3i \). Express \( \frac{z_1}{z_2} \) in the form \( a + ib \).
2. Express \( (1 + i)^6 \) in the form \( a + ib \).
3. Subtract \( 5e^{0.2i} \) from \( 6e^{2i} \). Express the answer in polar form.
4. Express the product \( (1 + 2i)(5 + 3i) \) in polar form.
5. Find the three roots of 1.

PART VI: Initial Value Problems
Find the solution.
1. \( y'' + 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 1 \)
2. \( \ddot{y} + y = 120te^{-t}, \quad y(0) = 0, \quad \dot{y}(0) = 12, \) and \( \dot{y} = \frac{dy}{dt} \)
1. Miscellaneous Math
An electrical cable must be laid to connect points A and B across a river, as shown in Figure 1. The ratio of costs of running the cable over water and land is \( k \). Find the location \( \frac{x}{d_1} \) that minimizes the cost of laying the cable.

![Diagram of cable laying](image)

Figure 1: Problem 1

2. Complex Variables

- Find all solutions of the equation \( z^6 + a^2 = 0 \) for \( z \). Assume \( a \neq 0 \). Represent the solutions graphically.
- Calculate \( 10e^{2i} - e^{-2i} \) in rectangular coordinates (real/imaginary form). Do not use a calculator to find the answer directly.
- Calculate \( \cosh(z_1) + \sinh(z_2) \), with \( z_1 = 2 - 2i \) and \( z_2 = 1 + i \). Do not use a calculator to find the answer directly.

3. Ordinary Differential Equations
A. Certain biological species has a birth rate which is proportional to the population with proportionality constant 3. The death rate is also proportional to the population, with proportionality constant 2. Suppose the population is 1000 at time zero. Determine:

1. Whether the population will be extinguished. If so, find the time for extinction.
2. If the population will not be extinguished, find the time for doubling.
B. Now suppose the birth rate is proportional to the square of the population with proportionality constant 0.3, while the death rate is proportional to the population, with proportionality constant 3. The initial population is 100. Find an expression for \( P(t) \) and describe whether the population is increasing or decreasing at \( t = 0 \).

C. Solve for \( y(x) \):

\[
\frac{dy}{dx} + y \sin(x) = y^2(x^2 + 1)
\]

4. Linear Algebra

A. Find the eigenvalues of the following matrix:

\[
\begin{bmatrix}
1 & 4 & a \\
0 & b & 1 \\
0 & 0 & 2
\end{bmatrix}
\]

(1)

B. Find the inverse of the above matrix assuming \( b \neq 0 \).

C. Does the system of equations \( Ax = b \) with

\[
A = \begin{bmatrix}
1 & 2 \\
4 & -2 \\
3 & -1 \\
7 & 1
\end{bmatrix}
\]

(2)

and \( b = [-9 \ 24 \ 15 \ 15]^T \) have a solution for \( x \)?

5. Vector Analysis

1. Find any vector tangent to the sphere \( x^2 + y^2 + z^2 = 1 \) at \((0.5, 0.25, 0.8292)\).

2. Find a unit vector which is normal to the plane \( x + 2y - z + 4 = 0 \).

3. Find the divergence of the vector function \( F \) given in cylindrical coordinates

\[
F(\rho, \theta, z) = (\rho^2 + \sin(\theta))\epsilon_\rho + (z \cos(\theta) - \rho)\epsilon_\theta + (z^2 \sin(2\theta))\epsilon_z
\]

where the unit vectors \( \epsilon_\rho, \epsilon_\theta \) and \( \epsilon_z \) are along the radial, tangential and vertical \( (z) \) directions.

6. Series and Sums

1. Find the first three terms of the MacLaurin expansion of \( f(x) = \frac{x^1 \cos(x)}{x^2 + 1} \).

2. Find the sum

\[
\sum_{k=3}^\infty e^{-2k}
\]

3. Expand the function \( f(x, y) = xy \sin(x) - \cos^2(xy) \) in a Taylor series around \((\pi, -\pi)\), retaining terms up to second power.

4. Is the following sum convergent?

\[
\sum_{n=1}^\infty (-1)^n \frac{1}{n(n^{1/3} + 2)}
\]
1. Miscellaneous Math

The total cost is \( w \sqrt{d_1^2 + x^2} + \lambda (d_2 - x), 0 \leq x \leq d_2 \),

where \( w \) is the cost over water and \( \lambda \) the cost over land.

\[
J(x) = w \sqrt{d_1^2 + x^2} + \lambda (d_2 - x)
\]

To minimize the cost we make \( \frac{dJ(x)}{dx} = 0 \)

so \( \frac{1}{2} w (d_1^2 + x^2)^{-1/2} \cdot 2x - \lambda = 0 \)

\[
\frac{w x}{\sqrt{d_1^2 + x^2}} = \lambda
\]

or \( \frac{x}{\sqrt{d_1^2 + x^2}} = \frac{\lambda}{w} = \frac{1}{K} \)

Solving \( \frac{x^2}{d_1^2 + x^2} = \frac{1}{K^2} \)

if \( K > 1 \): \( x = \frac{d_1/K}{\sqrt{1 - \frac{1}{K^2}}} = \frac{d_1/K}{\sqrt{K^2 - 1}/K} = \frac{d_1}{\sqrt{K^2 - 1}} \)
When $K \to \infty$ (very expensive to run cable over water in comparison to land) we get $x = 0$, which makes sense.

When $K \to 1$, $x$ grows. In this case, the minimum is obtained at the end of the interval: $x = d_2$. The same for $K^2 < 1$.

Therefore, to minimize the cost pick \[
\begin{align*}
X &= \frac{d_1}{\sqrt{K^2 - 1}} \\
\text{if } K > 1 \\
\text{otherwise pick } x &= d_2
\end{align*}
\]

Graphically: \[
\frac{1}{K} J(x) = K \sqrt{d_1^2 + x^2} + d_2 - x
\]

For $K > 1$:

\[\frac{1}{K} J(x) \Delta d_2 \quad x^* \quad d_2 \quad d_2 - x \quad d_2 \quad d_2 - x \quad Kd_1 \quad Kd_1 \]

For $K \leq 1$:
3. Ordinary Differential Equations

A. \[ \frac{dp}{dt} = 3P - 2P = P \]
then \[ P(t) = P_0 e^t, \] with \( P_0 = 1000 \).

1. The population increases exponentially.

2. \[ 1000 e^t = 2000 \]
\[ e^t = 2, \quad t = \ln 2 \]

B. \[ \frac{dp}{dt} = 0.3P^2 - 3P \]
\[ \frac{dp}{P(0.3P-3)} = dt \]
\[ \int_{P_0}^{P} \frac{dp}{P(0.3P-3)} = t \]

From integral table, \[ \int \frac{dx}{x(a+bx)} = -\frac{1}{b} \ln \left| \frac{a+bx}{x} \right| \]

use \( a = 0.3, \ b = -3 \)

then \[ \left[ \frac{1}{3} \ln \left( \frac{0.3P-3}{P} \right) \right]_{P_0}^{P} = t \]
When \(0.3P > 3\) we have \(P > 10\) and

\[
\ln \left( \frac{0.3P - 3}{P} \right) \bigg|_{P_0}^P = t
\]

So

\[
\ln \left( \frac{0.3P - 3}{P} \right) \bigg|_{P_0}^P = 3t \rightarrow \ln \left( \frac{0.3P - 3}{P} \right) - \ln \left( \frac{0.3P_0 - 3}{P_0} \right) = 3t
\]

\[
\ln \left[ \frac{0.3P - 3}{0.3P_0 - 3} \frac{P}{P_0} \right] = 3t
\]

\[
0.3P = \left( \frac{0.3P - 3}{P_0} \right)^{3t} P_0^t
\]

\[
P(t) = \frac{3}{0.3^t - \left( \frac{0.3P - 3}{P_0} \right)e^{3t}} \quad ; \quad P > 10
\]

Since \(P_0 = 100 > 10\), the above formula can be used.

At \(t = 0\)

\[
\frac{dP}{dt} = 0.3P^2 - 3P_0 = 0.3 \times 100^2 - 300 = 2700 > 0
\]

The population is increasing.
C. This is a Bernoulli equation of the form

\[ y' + y \frac{f(x)}{a} = y^a \frac{g(x)}{a} \]

with \( f(x) = \sin x \)
\( g(x) = x^2 + 1 \); \( a = 2 \)

The change of variables \( z = \frac{1}{y} \) puts the equation in the standard form:

\[ z' + (1-a) z f(x) = (1-a) g(x) \]

In this case we get

\[ z' - z \sin x = -(x^2 + 1) \]

The solution is

\[ z(x) = e^{-A(x)} \left( \int e^{A(x)} (-\int e^{A(x)} (x^2 + 1) \, dx + C) \right) \]

where \( A(x) = \int -\sin x \, dx = -\cos x \)

\[ z(x) = e^{-\cos x} \left( -\int e^{\cos x} (x^2 + 1) \, dx + C \right) \]

\[ y(x) = \frac{1}{z(x)} \]
2. Complex variables

a) $z^6 + a^2 = 0, \quad a \neq 0, \quad a \text{ real}$

The solutions will form a hexagon in the complex plane.

$z^6 = -a^2 \quad \rightarrow \quad z = (-a^2)^{\frac{1}{6}} = \pm \sqrt[6]{a^2} \mathrm{i}$

The other solutions are found by forming a symmetric hexagon:

\[
\begin{align*}
\text{Solutions} & = \pm \sqrt[6]{a^2} \mathrm{i} \\
& \pm \frac{\sqrt{3}}{2} \sqrt[6]{a^2} \pm \frac{1}{2} \sqrt[6]{a^2} \mathrm{i}
\end{align*}
\]

b) $e^{2\mathrm{i}} = \cos 2 + \mathrm{i} \sin 2$

$e^{-2\mathrm{i}} = \cos 2 - \mathrm{i} \sin 2$

$10e^{2\mathrm{i}} - e^{-2\mathrm{i}} = 2 \cos 2 + 8 \mathrm{i} \sin 2$

d) $\cosh z_1 = \frac{e^z + e^{-z}}{2} = \frac{e^{2\mathrm{i}} + e^{-2\mathrm{i}}}{2} = \frac{e^{(e^{2\mathrm{i})} + e^{-2\mathrm{i}})}{2}$

$\sinh z_2 = \frac{e^z - e^{-z}}{2} = \frac{e^{4\mathrm{i}} - e^{-4\mathrm{i}}}{2} = \frac{e(\mathrm{i}) - e^{-1}(\mathrm{i})}{2}$
\[
\cosh z_1 = \frac{e^2(\cos 2 - i \sin 2) + e^{-2}(\cos 2 + i \sin 2)}{2}
\]
\[
\cosh z_1 = \frac{e^2 \cos 2 + e^{-2} \cos 2 + (e^{-2} \sin 2 - e^2 \sin 2)i}{2}.
\]
\[
\cosh z_1 = \frac{\cos 2 (e^2 + e^{-2})}{2} + \frac{\sin 2 (e^{-2} - e^2)}{2} i.
\]
\[
\cosh z_1 = \cos 2 \cosh 2 - (\sin 2 \sinh 2)i.
\]
\[
\sinh z_2 = \frac{e(\cos 1 + i \sin 1) - e^{-1}(\cos 1 - i \sin 1)}{2}
\]
\[
\sinh z_2 = \frac{\cos 1 (e - e^{-1})}{2} + \frac{\sin 1 (e + e^{-1})}{2} i.
\]
\[
\sinh z_2 = \cos 1 \sinh 1 + (\sin 1 \cosh 1) i
\]
\[
\cosh z_2 + \sinh z_2 = \cos 2 \cosh 2 + \cos 1 \sinh 1
\]
\[+ (\sin 1 \cosh 1 - \sin 2 \sinh 2)i.\]
4. Linear Algebra

A. $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$ gives the eigenvalues.

$$
\begin{vmatrix}
\lambda - 1 & -4 & -a \\
0 & \lambda - b & -1 \\
0 & 0 & \lambda - 2
\end{vmatrix} = (\lambda - 1)(\lambda - b)(\lambda - 2)
$$

The eigenvalues are $\lambda = \frac{1}{2}$

B. Inverse: $\{A^{-1}\}_{(i,j)} = \frac{|M_{j,i}|(-1)^{i+j}}{\det A}$

$M_{11} = \begin{bmatrix} b & 1 \\ 0 & 2 \end{bmatrix}$, $|M_{11}| = 2b$

$M_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $|M_{12}| = 0$

$M_{13} = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$, $|M_{13}| = 0$

$M_{21} = \begin{bmatrix} 4 & a \\ 0 & 2 \end{bmatrix}$, $|M_{21}| = -8$

$M_{22} = \begin{bmatrix} 1 & a \\ 0 & 2 \end{bmatrix}$, $|M_{22}| = 2$
\[ M_{22} = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}, \quad |M_{22}| = 0 \]

\[ M_{31} = \begin{bmatrix} 4 & a \\ b & 1 \end{bmatrix}, \quad |M_{31}| = 4 - ba \]

\[ M_{32} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \quad |M_{32}| = -1 \]

\[ M_{33} = \begin{bmatrix} 1 & 4 \\ 0 & b \end{bmatrix}, \quad |M_{33}| = b \]

\[ \text{det } A = 2b \]

Then

\[ A^{-1} = \begin{bmatrix} 2b & -8 & 4 - ba \\ 0 & 2 & -1 \\ 0 & 0 & b \end{bmatrix} \times \frac{1}{2b} \]

\[ A^{-1} = \begin{bmatrix} 1 & -4/b \\ 0 & 1/b \end{bmatrix} \begin{bmatrix} 2b - \frac{a}{2} \\ \frac{2}{b} - \frac{a}{2} \\ \frac{1}{b} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1/2 \end{bmatrix} \]
C. If a solution exists, we need 

\[ \text{rank}(A) \geq \text{rank}([A|b]) \]

To find \( \text{rank}(A) \), we row operations.

\[
\begin{bmatrix}
1 & 2 \\
4 & -2 \\
3 & -1 \\
2 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 \\
4 & -2 \\
3 & -1 \\
2 & 1
\end{bmatrix} \rightarrow 
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

Columns are clearly

linearly independent \( \rightarrow \) \( \text{rank}(A) = 2 \)

\[
\begin{bmatrix}
2 & -9 \\
-2 & 24 \\
3 & -1 \\
7 & 1
\end{bmatrix}
\]

Augmented matrix \([A|b]\)

\[
\begin{bmatrix}
1 & 2 & -9 \\
1 & 0 & 24 \\
3 & -1 & 15 \\
4 & 2 & 0
\end{bmatrix} \times \frac{1}{2}
\]
\[
\begin{bmatrix}
1 & 2 & -9 \\
4 & -2 & 24 \\
5 & 0 & 15 \\
4 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 & -9 \\
8 & 0 & 24 \\
5 & 0 & 15 \\
4 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-3 & 0 & -9 \\
8 & 0 & 24 \\
5 & 0 & 15 \\
4 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-3 & 0 & -9 \\
5 & 0 & 15 \\
5 & 0 & 15 \\
4 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-3 & 0 & -9 \\
0 & 0 & 0 \\
5 & 0 & 15 \\
5 & 2 & 0
\end{bmatrix}
\times \frac{3}{5} = [3 \ 0 \ 9]
\]

We see that
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
5 & 0 & 15 \\
4 & 2 & 0
\end{bmatrix}
\frac{1}{3} c_3 + 2 c_2 = c_1
\]

Therefore \( \text{rank}(A|b) = 2 \)

\[
\Rightarrow \text{ solutions do exist.}
\]
5. Vector Analysis.

1. In spherical coordinates

\[ e_\theta = -\hat{x}\sin\theta + \hat{z}\cos\theta \]

This vector is tangent to the sphere and parallel to the \( xy \) plane.

For the point given: \( \psi = \arccos \left( \frac{2}{\sqrt{3^2 + 1^2 + 2^2}} \right) \)

\[ \theta = \arctan \left( \frac{2}{1} \right) \]

\( \psi = \arccos (0.8298) \)

\[ \theta = \arctan \left( \frac{0.25}{0.5} \right) = \arctan \left( \frac{1}{2} \right) \]

\[ \sin \theta = \frac{\sqrt{5}}{5} \]

\[ \cos \theta = \frac{2\sqrt{5}}{5} \]

so a tangent vector could be \( e_\theta = \begin{bmatrix} -\sqrt{5}/5 \\ 2\sqrt{5}/5 \\ 0 \end{bmatrix} \)
2. The gradient gives the desired direction:

\[ \nabla = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} , \quad \| \nabla \| = \sqrt{1 + 4 + 1} = \sqrt{6} \]

The desired vector is:

\[ \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{6}}{3} \end{bmatrix} \]

3. In cylindrical coordinates

\[ \text{div} \, F = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho F_\rho \right) + \frac{1}{\rho} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z} \]

\[ \text{div} \, F = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho (\rho^2 + \sin \theta) \right) + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left( 2 \cos \theta \rho - \rho \sin \theta \right) \]

\[ + \frac{\partial}{\partial z} \left( z \sin 2\theta \sin \phi \right) \]

\[ \text{div} \, F = \frac{1}{\rho} (\rho^2 + \sin \theta) + \frac{1}{\rho} (\rho \sin \theta) + 2 \rho \sin \phi \]

\[ \text{out} \, F = \frac{1}{\rho} (\rho^2 + \sin \theta (1 - 2)) + 2 \rho \sin \phi \]
6. Series and Sums

\[ f(x) \approx f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 \]

\[ f(0) = 0 \]

\[ f'(0) = \frac{(x+1)^2 [3\cos x - x^2 \sin x] - 2x(x^2 \cos x)}{(x^2+1)^2} \]

\[ f''(0) = \frac{(x+1)^2 [2x(\cdot) + (x^2+1)(2\cos x - 2x\sin x - 2x\sin x - x^2 \cos x)] - 2(x^2+1) \cdot 2x \cdot [\cdots]}{(x^2+1)^4} \]

\[ f'''(0) = \frac{2}{1} = 2 \]

Then, to second order,

\[ f(x) \approx x^2, \text{ near } x = 0 \]
2. \[ \sum_{k=3}^{\infty} e^{-2k} = \sum_{k=0}^{\infty} \left( \frac{1}{e^2} \right)^k - 1 - \frac{1}{e^2} - \frac{1}{e^4} \]

We know \[ \sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \] if \( |a| < 1 \)

since \( a = \frac{1}{e^2} < 1 \), we use the formula to obtain:

\[ \sum_{k=3}^{\infty} e^{-2k} = \frac{1}{1-\frac{1}{e^2}} - 1 - \frac{1}{e^2} - \frac{1}{e^4} \]

3. \( f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2} [f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2] \)

\[ f(a,b) = -\pi^2 \sin^2(\pi) - \cos^2(-\pi^2) = -\cos^2(\pi^2) = -0.8198 \]

\[ f_x = y \left( \sin x - x \cos x \right) + 2y \cos x \sin x y \sin x y \]

\[ f_{xy} = \sin x - x \cos x + 2 \cos x y \sin x y + y \left( 2x \sin x y + x \cos^2 x y \right) \]

\[ f_{xx} = y \left( \cos x - \cos x + x \sin x \right) + 2y \left( -y \sin^2 x y + y \cos^2 x y \right) \]

\[ f_y = x \sin x + 2x \cos x y \sin x y \]
\[
\frac{f_{yy}}{x} = 2x \left( -x \sin^2 x + x \cos^2 y \right)
\]

\[
\frac{f_x(a,b)}{x} = -\pi \left( \sin \pi - \pi \cos \pi \right) - 2\pi \cos(-\pi^2) \sin(-\pi^2) = -7.4290
\]

\[
\frac{f_{xy}(a,b)}{xy} = \sin \pi \pi - \pi \cos \pi + 2 \left[ \cos(-\pi^2) \sin(-\pi^2) - \pi \left( -\pi \sin^2(\pi^2) + \pi \cos^2(\pi^2) \right) \right]
\]

\[
\frac{f_{xy}(a,b)}{xy} = \pi^2 + 2 \left[ -\cos(\pi^2) \sin(\pi^2) + \pi^2 \sin(\pi^2) - \pi^2 \cos(\pi^2) \right]
\]

\[
\frac{f_{xy}(a,b)}{xy} = \pi^2 - 2 \sin(\pi^2) \cos(\pi^2) - 2\pi^2 = -17.3745
\]

\[
\frac{f_{xx}(a,b)}{xx} = -\pi^2 \sin(\pi) - 2 \pi \left( \pi \sin^2(-\pi^2) - \pi \cos^2(-\pi^2) \right)
\]

\[
\frac{f_{xx}(a,b)}{xx} = 2\pi^2 \sin^2(\pi^2) + 2\pi^2 \cos(\pi^2) = 2\pi^2 = 19.73
\]

\[
\frac{f_{yy}(a,b)}{yy} = \pi \sin \pi + 2\pi \cos(-\pi^2) \sin(-\pi^2)
\]

\[
\frac{f_{yy}(a,b)}{yy} = -2\pi \cos(\pi^2) \sin(\pi^2) = -2.4406
\]

\[
\frac{f_{yy}(a,b)}{yy} = 2\pi \left( -\pi \sin^2(-\pi^2) + \pi \cos^2(-\pi^2) \right)
\]

\[
\frac{f_{yy}(a,b)}{yy} = 2\pi^2 = 19.7392
\]
Then \( f(x, y) = -0.848 - 7.9290(x - \pi) - 2.4406(y + \pi) + 9.8696(x - \pi)^2 - 17.3745(x - \pi)(y + \pi) + 9.8696(y + \pi) \)

4. \[ \sum_{n=1}^{\infty} (-1)^n \frac{1}{n(n^{\frac{1}{3}} + 2)} \]

According to the Leibniz test, if \( \{a_n\} \) is decreasing with \( a_n \to 0 \) as \( n \to \infty \) then
\[ \sum_{n=1}^{\infty} (-1)^n a_n \] converges.

Here \( a_n = \frac{1}{n(n^{\frac{1}{3}} + 2)} \) is decreasing and \( a_n \to 0 \) as \( n \to \infty \):

\[ \{a_n\} = \left\{ \frac{1}{3}, \frac{1}{2(3^{\frac{1}{3}} + 2)}, \frac{1}{3(3^{\frac{2}{3}} + 2)}, \ldots \right\} \]

Thus \[ \sum_{n=1}^{\infty} (-1)^n \frac{1}{n(n^{\frac{1}{3}} + 2)} \] converges.
1. Miscellaneous Math
A cylindrical can is to hold 2π m³. The material for the top and bottom costs $1/m² and the material for the side costs $0.8/m². Find the radius, height and cost of the most economical can.

2. Complex Variables
- Use the deMoivre formula to derive a trigonometric identity for cos(3θ) in terms of sinθ and cosθ (no complex coefficients in the identity).
- Find all solutions of the equation z⁵ + a⁴ = 0 for z, assuming a ≠ 0. Represent the solutions graphically.
- Calculate tanh(z₁) + sinh(z₁z₂) in rectangular coordinates (real/imaginary form). Do not use a calculator to find the answer directly. Take z₁ = 1 + 2i and z₂ = 2e⁻ˣ⁺iy

3. Ordinary Differential Equations
A. The rate-of-change of the population density of bald eagles equals kN(1 - N/C), where N is the population density in individuals per km², k is a growth rate having units of years⁻¹ and C is the habitat’s capacity (same units as N). Suppose for bald eagles in a certain area, the capacity of the habitat is estimated to be 14.2 eagles/km², and k = 0.40 years⁻¹. A survey done in 1998 shows that the population density was 3.1 eagles/km². What do you estimate the population be in 2010 if environmental conditions remain the same, i.e. that the carrying capacity, C, and the growth rate, k, remain unchanged?

B. Solve for y(x):
\[ \frac{dy}{dx} + y \tan(x) = y^2(x - 1) \]
4. Linear Algebra
A. Find the eigenvalues and a set of eigenvectors for the following matrix:

\[
\begin{bmatrix}
a & 1 & 0 \\
0 & 1 & b \\
0 & -1 & 1 \\
\end{bmatrix}
\]  

(1)

B. Find the inverse of the above matrix assuming \( b \neq -1, a \neq 0 \)

C. Does the system of equations \( Ax = b \) with

\[
A = \begin{bmatrix}
2 & 1 \\
-3 & 4 \\
1 & 1 \\
1 & -1 \\
\end{bmatrix}
\]

(2)

and \( b = [1 \ 1 \ 1 \ -1]^T \) have a solution for \( x \)? If yes, find a solution. If no, state why.

5. Vector Analysis

1. Find a unit vector tangent to the surface \( x + y + z^2 = 1 \) at \( (1, -1, 1) \)

2. Find a unit vector normal to the same surface at the same point.

3. Find the curl of the vector function \( F \) given in cylindrical coordinates

\[ F(\rho, \theta, z) = (\rho^2 + \sin(\theta))e_\rho + (z \cos(\theta) - \rho) e_\theta + (z^2 \sin(2\theta))e_z \]

where the unit vectors \( e_\rho, e_\theta \) and \( e_z \) are along the radial, tangential and vertical \( z \) directions.

6. Series and Sums

1. Find the first three terms of the MacLaurin expansion of \( f(x) = \frac{x^3 \sin(x)}{x+1} \).

2. Find the sum

\[
\sum_{k=2}^{\infty} e^{-3k}
\]

3. Expand the function \( f(x, y) = xy \sin^2(xy) - y^2 \cos(y) \) in a Taylor series around \( (-\pi, \pi) \), retaining terms up to third power.
1) Miscellaneous math.

Cylinder dimensions

Volume = \( \pi r^2 h = 2\pi \, \text{m}^3 \)

Area of top and bottom, \( A_1 = 2\pi r^2 \) (\( \in \text{\#1/m}^2 \))

Area of side, \( A_2 = 2\pi rh \) (\( \in \text{\#0.3/m}^2 \))

Cost = \( C = 2\pi r^2 + 1.6\pi rh \)

There are 2 ways to proceed from this point:

a) Solve for one of the variables (\( r \) or \( h \)) from the volume requirement, substitute into the cost and minimize.

b) Use the method of Lagrange multipliers to solve a constrained optimization problem.
Solution a): \[ h = \frac{2}{r^2} \]

Substituting: \[ C = 2\pi r^2 + 1.6\pi \times \frac{x^2}{r^2} \]

\[ C(r) = 2\pi r^2 + \frac{3.2\pi}{r} \]

Differentiate:
\[
\frac{dc}{dr} = 4\pi r + \frac{3.2\pi}{r^2} = 0
\]

\[
\frac{4r^3 - 3.2}{r^2} = 0
\]

\[ \rightarrow r = \sqrt[3]{\frac{3.2}{4}} \]

\[ h = \frac{2}{\left(\frac{3.2}{4}\right)^{3/2}} \]

\[ C = 2\pi \left(\frac{3.2}{4}\right)^{2/3} + \frac{3.2\pi}{\sqrt{3.2/4}} \]

\[ r \approx 0.9293 \text{ m} \]

\[ h \approx 2.32 \text{ m} \]

\[ C \approx \$16.24 \]
solution b)

Lagrangian: \[ L = 2\pi r^2 + 1.6\pi rh + \lambda (\pi r^2 h - 2\pi) \]

Optimality conditions:
\[
\begin{align*}
\frac{\partial L}{\partial r} &= 0 \\
\frac{\partial L}{\partial h} &= 0 \\
\frac{\partial L}{\partial \lambda} &= 0
\end{align*}
\]

\[
\frac{\partial L}{\partial r} = 4\pi r + 1.6\pi h + 2\lambda \pi rh = 0 \quad \cdots (1)
\]

\[
\frac{\partial L}{\partial h} = 1.6\pi r + 2\lambda \pi r^2 = 0 \quad \cdots (2)
\]

\[
\frac{\partial L}{\partial \lambda} = \pi r^2 h - 2\pi = 0 \quad \cdots (3)
\]

From (2) \[ 1.6 + \lambda \pi r = 0 \]
\[ \lambda r = -1.6 \]

Into (1):
\[
4\pi r + 1.6\pi h - 3.2\pi h = 0
\]
\[ 4r = 1.6h \]
\[ r = 0.4h \]

Using (3):
\[ \pi (0.4h)^2 h = 2\pi \]
\[ h = 2\pi \]

2. Complex Variables

a) de Moivre's Formula: \((\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\).

Take \(n = 3\)

\[(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta\]

\[\cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta + i \sin^3 \theta\]

\[= \cos 3\theta + i \sin 3\theta\]

\[\left(\cos^3 \theta - 3 \cos \theta \sin^2 \theta\right) + i \left(3 \cos^2 \theta \sin \theta - \sin^3 \theta\right) = \cos 3\theta + i \sin 3\theta\]

Matching real parts:

\[\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta\]

b) \(z^5 = -a^4\). Since 5 is odd, one root will be \(-a^{4/5}\). The roots are on the vertices of a regular pentagon in the complex plane.
\[ \begin{align*}
Z_1 &= -a e^{\frac{\pi i}{5}} \\
Z_2 &= a e^{\frac{\pi i}{5} (108^\circ)} \\
Z_3 &= a e^{\frac{\pi i}{5} (18^\circ)} \\
Z_4 &= a e^{\frac{\pi i}{5} (-18^\circ)} \\
Z_5 &= a e^{\frac{\pi i}{5} (108^\circ)} \\
\end{align*} \]

\[ \tanh Z_1 = \frac{e^{Z_1} - e^{-Z_1}}{e^{Z_1} + e^{-Z_1}} \]

\[ \sinh Z_1Z_2 = \frac{1}{2} \left( e^{Z_1Z_2} - e^{-Z_1Z_2} \right) \]

\[ Z_1 = 1 + 2i \quad Z_2 = 2 e^{-\frac{\pi i}{6}} \]

\[ e^{Z_1} = e^{(\cos 2 + is\sin 2)} = e^{1+2i} = e e^{2i} \]

\[ e^{-Z_1} = e^{-1(\cos 2 - is\sin 2)} = -1 e^{-2i} = -1 e^{-2i} \]

\[ \tanh Z_1 = \frac{e^{(\cos 2 + is\sin 2)} - e^{-1(\cos 2 - is\sin 2)}}{e^{(\cos 2 + is\sin 2)} + e^{-1(\cos 2 - is\sin 2)}} \]
\[
\tanh z_1 = \frac{(e - e^{-1}) \cos 2 + (e + e^{-1}) i \sin 2}{(e + e^{-1}) \cos 2 + (e - e^{-1}) i \sin 2}
\]

Multiply by the conjugate of the denominator:

\[
\tanh z_1 = \frac{[(e - e^{-1}) \cos 2 + (e + e^{-1}) i \sin 2] [(e + e^{-1}) \cos 2 - (e - e^{-1}) i \sin 2]}{(e + e^{-1})^2 \cos^2 2 + (e - e^{-1})^2 \sin^2 2}
\]

\[
\tanh z_1 = \frac{(e - e^{-1})(e + e^{-1}) + [(e + e^{-1})^2 - (e - e^{-1})^2] \sin 2 \cos 2}{(e + e^{-1})^2 \cos^2 2 + (e - e^{-1})^2 \sin^2 2}
\]

\[
\tanh z_1 = \frac{(e - e^{-1})(e + e^{-1})}{(e + e^{-1})^2 \cos^2 2 + (e - e^{-1})^2 \sin^2 2} + \frac{(e + e^{-1})^2 - (e - e^{-1})^2 \sin 2 \cos 2}{(e + e^{-1})^2 \cos^2 2 + (e - e^{-1})^2 \sin^2 2}
\]

Real

Imag

These can be evaluated without the need of a calculator with complex functions.

Now

\[
e^{z_1 z_2} = e^{1 + 2i} = e^{2i} e^{-\pi i/6}
\]

\[
= 2 e^{(2 - \pi i/6)i} = 2 e^{(\cos (2 - \pi i/6) + i \sin (2 - \pi i/6))}
\]

and

\[
e^{-z_1 z_2} = 2 e^{-1} (\cos (2 + \pi i/6) - i \sin (2 + \pi i/6))
\]
\[
\sinh z_2 = \frac{1}{2} \left( 2e^{\cos (2-\pi/6)} + i \sin (2-\pi/6) \right) - 2e^{-i} (\cos (2+\pi/6) - i \sin (2+\pi/6))
\]

\[
\sinh z_2 = \begin{pmatrix}
\cos (2-\pi/6) - e^{-i} \cos (2+\pi/6) \\
\sin (2-\pi/6) + e^{-i} \sin (2+\pi/6)
\end{pmatrix}
\]

The desired quantity is then

\[
\text{tanh} z_1 + \sinh z_1 z_2 = \text{Real} + i \text{Imaginary}
\]

\[
\text{Real} = \cos (2-\pi/6) - e^{-i} \cos (2+\pi/6) + \frac{(e^{-i})(e^{i}e^{-i})}{(e^{i})^2 \cos^2 (2) + (e^{-i})^2 \sin^2 (2)}
\]

\[
\text{Imaginary} = \sin (2-\pi/6) + e^{-i} \sin (2+\pi/6) + \frac{(e^{i})^2 (e^{-i})^2}{(e^{i})^2 \cos^2 (2) + (e^{-i})^2 \sin^2 (2)}
\]

Numerically:

\[
\text{tanh} z_1 + \sinh z_1 z_2 = -15.094 + 12.85i
\]
\[
\frac{dN}{dt} = kN \left(1 - \frac{N}{C}\right) = kN - \frac{KN^2}{C}
\]

\[
\frac{dN}{kN - \frac{KN^2}{C}} = dt \quad \rightarrow \quad \int \frac{dN}{kN - \frac{KN^2}{C}} = t + \alpha
\]

Using a table of integrals:

\[
\int \frac{dx}{x(a+x+b)} = \frac{-1}{b} \ln \left| \frac{a+x+b}{x} \right|
\]

\[
\frac{1}{K} \int \frac{dN}{N \left(\frac{-N}{C} + 1\right)} = \frac{1}{K} \left[ -\ln \left| \frac{-N}{C} + 1 \right| \right]
\]

\[
\begin{cases}
    a = -\frac{1}{C} \\
    b = 1
\end{cases}
\]

so

\[
\frac{-1}{K} \ln \left| \frac{-N}{C} + 1 \right| = t + \alpha
\]

We know \( C = 14.2 \), \( k = 0.4 \) and \( N = 3.1 \) for \( t = 1998 \)

Then

\[
\alpha = -1998 - \frac{1}{0.4} \ln \left| \frac{-3.1}{14.2} + 1 \right| = -1994.556
\]

For \( t = 2010 \): we need to solve for \( N \):

\[
\ln \left| \frac{-N}{C} + 1 \right| = -k(t + \alpha)
\]

\[
\left| \frac{-N}{C} + 1 \right| = e^{-k(t+\alpha)}
\]

\[
\left| \frac{-N}{C} + 1 \right| = e
\]
Assuming \( 1 - \frac{N}{c} > 0 \), \( N > 0 \) (need to be checked)

\[ \frac{-N + 1}{c} = -K(t + \alpha) \]

Then \( N = \frac{1}{e^{-K(t + \alpha)} + \frac{1}{C}} \)

For \( t = 2010 \): \( N = 13.79 \text{ eagles/km}^2 \)

---

4. Linear Algebra

A. Eigenvalues: \( |\lambda I - A| = 0 \)

\[
\det \begin{bmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda
\end{bmatrix} - \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
\lambda - a & -1 & 0 & \lambda - a \\
0 & \lambda - b & -1 & 0 \\
0 & 0 & \lambda - 1 & \lambda - 1 \\
-1 & 0 & \lambda - 1 & 0
\end{bmatrix}
\]

\[ (\lambda - a)(\lambda - 1)^2 + b(\lambda - a) = 0 \]

Factor \( (\lambda - a) \): \( (\lambda - a)(\lambda - 1)^2 + b = 0 \)

Solutions: \( \lambda = a \)

\[ \lambda = 1 + \sqrt{-b} \]

\[ \lambda = 1 - \sqrt{-b} \]
Eigenvectors, \( \mathbf{A} \mathbf{V} = \lambda \mathbf{V} \)

- \[
\begin{bmatrix}
  a & 0 \\
  b & 1 \\
  0 & -1
\end{bmatrix}
\begin{bmatrix}
  v_1 \\
  v_2 \\
  v_3
\end{bmatrix} =
\begin{bmatrix}
  \lambda v_1 \\
  \lambda v_2 \\
  \lambda v_3
\end{bmatrix} =
\begin{bmatrix}
  av_1 + bv_2 \\
  v_2 + bv_3 \\
  -v_2 + v_3
\end{bmatrix}
\]

For \( \lambda = a \):
- \( av_1 + bv_2 = av_1 \rightarrow v_2 = 0 \)
- \( bv_3 = av_2 \rightarrow v_3 = 0 \)

The eigenvectors are defined only by direction: take \( v_1 = \) any non-zero vector.

For \( \lambda = 1 + \sqrt{-b} \):
- \[
\begin{cases}
  av_1 + bv_2 = (1 + \sqrt{-b})v_1 \\
  v_2 + bv_3 = (1 + \sqrt{-b})v_2 \\
  -v_2 + v_3 = (1 + \sqrt{-b})v_3
\end{cases}
\]
- Take \( v_1 = 1 \)
- \( v_2 = 1 + \sqrt{-b} - a \)
- \( v_3 = \frac{1 + \sqrt{-b} - a}{\sqrt{-b}} \)

For \( \lambda = 1 - \sqrt{-b} \): Take \( v_1 = 1 \)
- \( v_2 = 1 - \sqrt{-b} - a \)
- \( v_3 = \frac{1 + \sqrt{-b} - a}{\sqrt{-b}} \)

The eigenvalues/eigenvectors are:

- \( \lambda = a \): \( \lambda = 1 + \sqrt{-b} \)
- \( \lambda = 1 - \sqrt{-b} \)
\[ A^{-1} = \frac{T}{\det(A)} \]

**Cofactor matrix**

\[
A_{\text{cof}} = \begin{pmatrix}
(-1)^{1+1} M_{1,1} = (-1)^2 \begin{vmatrix} 1 & b \\ -1 & 1 \end{vmatrix} = 1+b \\
(-1)^{1+2} M_{1,2} = (-1)^3 \begin{vmatrix} 0 & b \\ 0 & 1 \end{vmatrix} = 0 \\
(-1)^{1+3} M_{1,3} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} = 0
\end{pmatrix}
\]

\[
A_{\text{cof}} = \begin{pmatrix}
(-1)^{2+1} \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = -1 \\
(-1)^{2+2} \begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} = a \\
(-1)^{2+3} \begin{vmatrix} a & 1 \\ 0 & -1 \end{vmatrix} = a \\
(-1)^{3+1} \begin{vmatrix} 1 & 0 \\ 1 & b \end{vmatrix} = b \\
(-1)^{3+2} \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = -ab \\
(-1)^{3+3} \begin{vmatrix} a & 1 \\ 0 & 1 \end{vmatrix} = a
\end{pmatrix}
\]

\[ A_{\text{cof}} = \begin{bmatrix}
1+b & 0 & 0 \\
-1 & a & a \\
b & -ab & a
\end{bmatrix}, \quad A_{\text{cof}}^T = \begin{bmatrix}
1+b & -1 & b \\
0 & a & -ab \\
0 & a & a
\end{bmatrix} \]
\[
\det A = \begin{vmatrix}
a & 1 & 0 & a \\
b & 0 & 1 & 0 \\
0 & -1 & 0 & -1 \\
-1 & 0 & a & 0 \\
\end{vmatrix}
\]

\[
\det A = a + ab = a(b+1), \quad b \neq -1, \ a \neq 0
\]

\[
\Rightarrow A^{-1} = \begin{bmatrix}
\frac{1}{a} & \frac{-1}{a(b+1)} & \frac{b}{a(b+1)} \\
0 & \frac{1}{b+1} & \frac{-b}{b+1} \\
0 & \frac{1}{b+1} & \frac{1}{b+1}
\end{bmatrix}
\]

C. A solution will exist only if \( \text{rank } A = \text{rank } ([A/b]) \)

To find \( \text{rank } A \), we put the matrix in row echelon form

\[
\begin{bmatrix}
2 & 1 \\
-3 & 4 \\
1 & 1 \\
1 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 1 \\
-3 & 4 \\
1 & 1 \\
2 & 0
\end{bmatrix}
\]

\[\leftarrow \text{ these two columns are clearly linearly independent (not parallel)}\]

\[
\Rightarrow \text{rank } A = 2
\]
Now take the augmented matrix:

\[
\begin{bmatrix}
2 & 1 & 1 \\
-3 & 4 & 1 \\
1 & 1 & 1 \\
1 & -1 & -1 \\
\end{bmatrix} + \begin{bmatrix}
2 & 1 & 1 \\
-3 & 4 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 1 & 1 \\
-3 & 4 & 1 \\
-2 & -3 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

the three column vectors are obviously linearly independent

\[\Rightarrow \text{rank}(A|b) = 3\]

\[\Rightarrow \text{rank } A \neq \text{rank } (A|b)\]

(no solution)
5. Vector Analysis

Parameterize the surface: let $x = t$

Since no specific direction is required, take the curve defined by intersecting the plane $x = y$ with the surface. This is done for simplicity, any other curve can be used, provided it passes through $(1, -1, 1)$

Curve: \[
\begin{align*}
    x &= t \\
    y &= -t \\
    z &= \sqrt{1 + t^2 - t}
\end{align*}
\]

Point: $t = 1$

Position vector \( \mathbf{r}(t) = (x(t), y(t), z(t)) \)

Tangent vector: $\mathbf{T} = \frac{d\mathbf{r}}{dt}$

\[
\mathbf{T} = \left(1, -1, \left. \frac{dz(t)}{dt} \right|_{t=1} \right)
\]

\[
\mathbf{T} = \left(1, -1, \frac{1}{2} (1+t^2-t)^{-\frac{1}{2}} \times (2t-1) \right|_{t=1}
\]

\[
\mathbf{T} = \left(1, -1, \frac{1}{2} \right)
\]

Unit vector

\[
\mathbf{T}^u = \left(\frac{1}{\sqrt{2.25}}, -\frac{1}{\sqrt{2.25}}, \frac{1}{2\sqrt{2.25}} \right)
\]
2. The normal vector can be found from the gradient:

\[ f(x, y, z) = x + y x + z^2 - 1 \]

\[ \nabla f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = [1 + y, x, 2z] \]

\[ \nabla f = \begin{bmatrix} 0, 1, 2 \end{bmatrix}, \quad ||\nabla f|| = \sqrt{5} \]

\[ \nabla f_{(\text{unit})} = (0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) . \]

3. In cylindrical coordinates:

\[ \text{Curl } F = \nabla \times F = \left( \frac{1}{\rho} \frac{\partial (\rho F_\theta)}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right) e_\rho + \left( \frac{\partial F_\theta}{\partial z} - \frac{\partial F_z}{\partial \phi} \right) e_\theta + \frac{1}{\rho} \left( \frac{\partial (\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi} \right) e_z \]
\[ \frac{\partial F_z}{\partial \theta} = 2z^2 \cos 2\theta; \quad \frac{\partial F_\theta}{\partial z} = \cos \theta \]

\[ \frac{\partial F_\theta}{\partial z} = 0; \quad \frac{\partial F_z}{\partial \phi} = 0 \]

\[ \frac{\partial (\rho F_\theta)}{\partial \phi} = \frac{\partial (yz \cos \theta - \rho^2)}{\partial \phi} = z \cos \theta - 2\rho \]

\[ \frac{\partial F_\theta}{\partial \theta} = \cos \theta \]

\[ \vec{\nabla} \times \vec{F} = \left( \frac{2z^2 \cos 2\theta - \cos \theta}{\rho} \right) \vec{e}_\phi \]

\[ + \frac{1}{\rho} \left( z \cos \theta - 2\rho - \cos \theta \right) \vec{e}_z \]
6. Series and Sums

**Maclaurin: Taylor on x=0**

\[ f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 \]

\[ f(0) = 0 \]

\[ f'(x) = \frac{(x+1)[3x^2\sin(x) + x^2\cos(x)] - x^3\sin(x)}{(x+1)^2} \]

\[ f'(0) = 0 \]

\[ f''(x) = \frac{(x+1)^2[6x^2\sin(x) + x^2\cos(x)] + (x+1)[2x(\cdot) + x^2(\cdot)]}{(x+1)^4} \]

\[ - 3x^2\sin(x) - x^3\cos(x) \]

\[ - 2(x+1) \left[ \cdots \right] \]

\[ f''(0) = 0 \]

The first three terms are 0, 0, 0.
2. \[
\sum_{k=2}^{\infty} e^{-3k} = \sum_{k=0}^{\infty} (e^{-3})^{k} - e^{-3x0} - e^{-3x1} \\
= \frac{1}{1 - e^{-3}} - 1 - e^{-3} \\
\]

3. \[f(x, y) = xy \sin^2 xy - y^2 \cos y\]

2-variable Taylor series:
\[
f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x} (x-x_0) + \frac{\partial f}{\partial y} (y-y_0) \\
+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (x-x_0)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} (y-y_0)^2 + \frac{\partial^2 f}{\partial x \partial y} (x-x_0)(y-y_0) \\
+ \ldots \] (meant to ask for \((-\pi, 0)\))

\[\frac{\partial f}{\partial x} = y (\sin^2 xy + 2xy \sin xy \cos xy)\]

\[\left. \frac{\partial f}{\partial x} \right|_{(-\pi, 0)} = 0\]
\[
\frac{2f}{\partial y} = x (\sin^2 xy + 2xysin xy \cos xy) - 2y \cos y + y^3 
\]

\[
\left. \frac{\partial f}{\partial y} \right|_{(-\pi, \theta)} = 0
\]
Problem (1) [Thermodynamics]

A Rankine cycle (shown in the Figure) with One Open Feed-Water-Heater (FWH) and the data given in the Table below.

**Determine (per one kg entering the turbine):**
1- Extraction Steam kg/kg, 
2- Turbine Work, kJ/kg
3- Heat input, kJ/kg
4- Cycle efficiency %

<table>
<thead>
<tr>
<th>State</th>
<th>P, kPa</th>
<th>T, C</th>
<th>X</th>
<th>h kJ/kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
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<td></td>
<td></td>
<td>192.3</td>
</tr>
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<td>400</td>
<td></td>
<td>0</td>
<td></td>
</tr>
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<td>608.8</td>
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<td>400</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
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<tr>
<td>7</td>
<td>10</td>
<td></td>
<td>0.816</td>
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</tbody>
</table>
**Problem (2) [Fluid Mechanics]**

A household faucet produces a jet that gets smaller in diameter as it approaches the sink (see Fig. below). At a certain flow, water coming out will fill an 8-oz glass in 8 s and produce a jet diameter at faucet exit of \( \frac{1}{2} \text{ in.} \). If the sink bottom is 8 in. below the faucet exit, what is the jet diameter at impact?

![Diagram of faucet and sink](image)

**Problem (3) [Control Systems]**

Determine if the closed-loop system is stable:

![Closed-loop system diagram](image)
Problem (4) [Solid Mechanics]

For the beam shown below, draw the shear force and bending moment diagrams. Find the magnitude and location of the maximum bending moment.

![Beam Diagram]

Problem (5) [Heat Transfer]

In a single-pass counterflow heat exchanger, 4536 kg/hr of water enter at 15°C and cool 9071 kg/hr of
an oil having specific heat of 2093 J/kg.°C from 93°C to 65°C. If the overall heat transfer coefficient is
284 W/m².°C, determine the surface area required.

**Assume:** Steady flow and Constant properties

![Heat Exchanger Diagram]

- Oil
  - \( \dot{m}_h = 9071 \text{ kg/hr} \)
  - \( T_{h,i} = 93°C \)
  - \( C_{p,h} = 2093 \text{ J/kg.°C} \)

- Water
  - \( \dot{m}_w = 4536 \text{ kg/hr} \)
  - \( T_{c,i} = 15°C \)
  - \( T_{h,o} = 65°C \)

- \( U = 284 \text{ W/m}^2.°\text{C} \)
1. (Thermodynamics)

(i) Identify each of the bullets on the figure below with the letter corresponding to the following states:

(a) Saturated vapor
(b) Compressed liquid
(c) Superheated vapor
(d) Saturated liquid
(e) Saturated liquid-vapor mixture
(f) Critical point

(ii) Write TRUE or FALSE (not T or F)
- A compressor is an example of a closed system
- For a cycle, the net work is necessarily zero.
- In a household refrigerator, the freezer compartment serves as the condenser
- Exergy is always being produced
- A quantity whose cyclic integral is zero is not a property.
- A throttling process can occur adiabatically

(iii) A heat engine has a thermal efficiency of 45%. How much power does the engine produce when heat is transferred into it at a rate of $10^9$ kJ/hr? Write your answer in space below

______________ MW
(iv) Which temperature-specific entropy state diagram below correctly represents the isobaric cooling process occurring in a heat exchanger? Circle the correct answer.

(v) A fuel is burned with 80 percent theoretical air. This is equivalent to (choose one)

(a) 20% excess air  
(b) 80% excess air  
(c) 20% deficiency of air  
(d) 80% deficiency of air  
(e) Stoichiometric amount of air

2. (Heat Transfer)
Steam at temperature of 500 C flows through a stainless steel (AISI1010 k = 56.5 W/m.K ) of 60-mm inside diameter and 75-mm outside diameter. The convection heat transfer coefficients are (inside and outside respectively) 1000 & 20 W/m² K. The surrounding is at 25 C.

Determine:
1- The heat loss to the surrounding (per unit cylinder length, q', W/m),
2- The inner surface temperature C,
3- The outer surface temperature C.

3. (Vibrations)
An electric motor is mounted on an elastic base as shown in the picture. The mass of the motor and base is M. The rotor is unbalanced. The unbalance is equivalent to a concentrated mass m at a radius r from the center. The motor rotates at w rad/sec. Using k=1000 N/m, m=0.1 kg, M=20 kg, r=0.01 m and w=11.2 rad/sec, find the amplitude of vertical vibrations of the assembly. Use the amplification chart below, where the horizontal axis is the forcing frequency to natural frequency ratio and the vertical axis is the ratio of the vibration amplitude times the effective spring constant (kX) to the magnitude of the forcing input Fo. Assume no damping.
3. Vibrations

The vertical force is $F_0 \sin(\omega t)$, with $F_0 = m \omega^2$.

The effective spring constant is $2k = 2000 \text{ N/m}$.

The frequency is $f = \frac{\omega}{2\pi} = \frac{11.2}{2\pi} = 1.7825 \text{ Hz}$.

The natural frequency is $f_0 = \sqrt{\frac{2k}{M}} = \sqrt{\frac{2000}{20}} = 10 \text{ rad/s}$.

So $f_0 = \frac{\omega_0}{2\pi} = \frac{1.7825}{1.5915}$.

So $\frac{f}{f_0} = \frac{1.7825}{1.5915} = 1.12$.

From the chart, for zero damping: $\frac{kx}{(f_0)} = 3.25$.

So $x = \frac{3.25 F_0}{2k} = \frac{3.25 \times 0.1 \times 10^2}{2000}$.

$$x = 0.00205 \text{ m}$$

$x = 2.05 \times 10^{-3} \text{ m} = 2.04 \text{ mm}$.
4. (Control Systems)

Reduce the following block diagram to find the transfer function $Y(s)/R(s)$:

5. (Fluid Mechanics)

Determine the pressure drop in the entrance region of the channel flow (see Figure below) assuming an entrance length equal to $\ell / h = 0.04$ Re. The entrance-velocity profile is uniform. The Reynolds number of the flow is 1500.

(Note: In a channel the maximum velocity at the centerline at the end of the entrance length where the velocity profile is parabolic is $u_{max} = (\frac{3}{2})V$ where $V$ is the average velocity.)
For the beam shown below, find an expression for the rotation at B and the deflection at B using Castigliano’s method.

**Rotation**

Origin

Moment

Limits

**Deflection**

Origin

Moment

Limits
Sample Problems for the Doctoral Qualifying Exam
submitted by
Asuquo B. Ebiana

CLOSED BOOK

FLUID MECHANICS

(1) Simplify the Navier-Stokes equations (given in vector form below) for flow in a horizontal, rectangular channel assuming all streamlines parallel to the walls (see Figure below). Let the x-direction be in the direction of flow.

\[ \text{Navier-Stokes equations: } \rho \frac{\text{DV}}{\text{Dt}} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V} \]

![Flow diagram](image)

HEAT TRANSFER

(1) At a given instant of time the temperature distribution within an infinite homogeneous body (see Figure below) is given by the function:

\[ T(x, y, z) = x^2 - 2y^2 + z^2 - xy + 2yz \]

Assuming constant properties and no internal heat generation, determine the regions where the temperature changes with time.

[Use the heat equation:

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \]

![Temperature distribution](image)
THERMODYNAMICS

(1) Water is being heated in a closed pan on top of a range while being stirred by a paddle wheel (see Figure below). During the process, 30 kJ of heat is transferred to the water, and 5 kJ of heat is lost to the surrounding air. The paddle-wheel works amounts to 500 N-m. Determine the final energy of the system if its initial energy is 10 kJ.

OPEN BOOK

MATH

(1) (a) Solve the equation $x^2y'' - xy' + \kappa^2y = 0$, where $\kappa$ is a real constant.

(b) For what values of $\kappa$ are there non trivial solutions (i.e. $\neq 0$) with $y(1) = y(a) = 0$?

FLUID MECHANICS

(1) The wind reaches a speed of 100 km/h in a storm. Calculate the force acting on a 1 m x 2 m window facing the storm. The window is in a high-rise building, so the wind speed is not reduced due to ground effects. Use $\rho = 1.2$ kg/m$^3$. Work with gage pressures.

HEAT TRANSFER

(1) Assume that a person can be approximated as a cylinder of 0.3-m diameter and 1.8-m height with a surface temperature of 24°C. Calculate the body heat loss while this person is subjected to a 15 m/s wind whose temperature is -5°C (see Figure below).
Assume: (1) Steady-state conditions, (2) Uniform surface temperature, (3) Negligible heat loss from cylinder top and bottom surfaces, (4) Negligible radiation effects.

**THERMODYNAMICS**

(1) During some actual expansion and compression processes in piston-cylinder devices, the gases have been observed to satisfy the relationship $PV^n = C$, where $n$ and $C$ are constants. Calculate the work done when a gas expands from a state of 150 kPa and 0.03 m$^3$ to final volume of 0.2 m$^3$ for the case of $n = 1.3$. 

1. (Thermodynamics) A refrigeration cycle including a heat exchanger operates with ideal air \((k=1.4, \ C_p = 0.24 \text{ Btu/lb}^\circ)\), as shown in Figure 1. The expander provides part of the mechanical work required to drive the compressor. Air enters the compressor at 14.7 psi and 0° F and leaves it at 80 psi. It enters the expander at 60 ° F. Calculate the coefficient of performance, assuming steady-state, steady-flow conditions. Remember that the coefficient of performance measures the useful heat extraction per unit of net mechanical work required to run the cycle.

Figure 1: Problem 1
2 (Control Systems) The following plant transfer function is operated under unity-feedback proportional control with gain $k$.

$$G(s) = \frac{s^2 + 0.5s + 4}{(s - 1)^2(s^2 + 10s + 100)(s + 2)}$$

Without solving the characteristic equation explicitly, determine if $k = 1000$ would result in closed-loop stability.

3 (Solid Mechanics/Design) Castigliano’s Theorem: If a elastic structure is subjected to point forces $P_i$, the deflection $x_i$ at the location and in the direction of $P_i$ is

$$x_i = \frac{\partial U}{\partial P_i}$$

where $U$ is the energy of deformation.

A beam under pure bending has energy

$$U = \int_0^L \frac{M^2}{2EI} \, dx$$

where $M(P_i, x)$ is the bending moment. Use this information to derive the formula for the tip deflection of a cantilever beam under the action of a transversal force at the tip: $\delta = \frac{FL^3}{3EI}$.

4 (Fluid Mechanics) An S-shaped sprinkler with arm radius $l$ rotates about its axis with no friction. Water flows with velocity $v$ relative to the tube of cross section $A$, as shown in Figure 2. Assuming that the water density is $\rho$ and that the moment of inertia of the sprinkler about the axis of rotation is $I$, find the angular acceleration of the sprinkler when the water starts to flow.

![Figure 2: Problem 4](image-url)
5 (Vibrations) Figure 3 shows a vibratory system with viscous friction between each block and the ground. Find the natural frequencies and determine if the system is underdamped. If affirmative, find the damping ratios. Take $k_1 = 1$, $k_2 = 2$, $m_1 = 2$, $m_2 = 1$, $b = 0.1$, $b_1 = 0.001$, $b_2 = 0.005$.

![Figure 3: Problem 5](image)

6 (Heat Transfer) Steam at temperature of 500 C flows through a stainless steel (AISI1010 k= 56.5 W/m.K ) of 60-mm inside diameter and 75-mm outside diameter. The convection heat transfer coefficients are (inside and outside respectively) 1000 and 20 W/m2 K. The surroundings are at 25 C. Determine:

1. The heat loss to the surrounding (per unit cylinder length, in W/m),
2. The inner surface temperature.
3. The outer surface temperature.
1) (Thermodynamics)

Find $T_2$: \[ \frac{T_2}{T_1} = \left( \frac{P_2}{P_1} \right)^{\frac{k-1}{k}} \]

\[ \Rightarrow T_2 = \left( \frac{30}{14.7} \right)^{0.286} \frac{T_1}{T_1} = 1.624 \text{ } T_1 = 747 \text{ °R} \]

The work of the compressor can be found as

\[ \omega_c = h_2 - h_1 = Cp(T_2 - T_1) = 0.24(747 - 460) = 68.5 \text{ Btu/lb} \]

The work of the turbine is

\[ \omega_t = h_3 - h_4 \text{ (need } T_4) \]

\[ \frac{T_3}{T_4} = \left( \frac{P_3}{P_4} \right)^{\frac{k-1}{k}}, \Rightarrow T_4 = 320 \text{ °R} \]

\[ \Rightarrow \omega_t = Cp(T_3 - T_4) = 48 \text{ Btu/lb} \]

The net work is \[ \omega_{net} = 20.9 \text{ Btu/lb} \]

The rejected heat is \[ \varphi_H = h_2 - h_3 = Cp(T_2 - T_3) = 54.5 \text{ Btu/lb} \]

The heat transferred from the turbine chamber is \[ \varphi_L = Cp(T_1 - T_4) = 33.6 \text{ Btu/lb} \]

The C.O.P. \[ \frac{\varphi_L}{\omega_{net}} = \frac{33.6}{20.9} = 1.61 \]
The characteristic equation is:

\[ 1 + Kg(s) = 0 \]

\[ 1 + \frac{k(s^2 + 0.5s + 4)}{(s-1)^2(s^2 + 10s + 100)(s+2)} \]

\[ (s-1)^2(s^2 + 10s + 100)(s+2) + 1000(s^2 + 0.5s + 4) = 0 \]

Expanding:

\[ s^5 + 10s^4 + 97s^3 + 972s^2 + 220s + 4200 = 0 \]

**Routh Table**:

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Sign change! No need to continue.

System will be unstable with \( k = 1000 \).
3) (Solid Mechanics)

Bending moment / Shear Force Diagrams.

\[ M(x, P) = -PL + P \lambda = P(x-L) \]

\[ U = \int_0^L \frac{M^2(x, P)}{2EI} \, dx \]

\[ U = \frac{1}{2EI} \int_0^L P(x-L)^2 \, dx = \frac{P^2}{2EI} \int_0^L (x-L)^2 \, dx \]

\[ U = \frac{P^2}{2EI} \left[ \frac{(x-L)^3}{3} \right]_0^L = \frac{P^2}{6EI} \left( 0^3 - (-L)^3 \right) \]
A water sprinkler consists of a tube bent into an s-shape as shown below. The axis of the sprinkler is vertical and the s-shaped tube can rotate about the axis without friction. The water in the tube flows with speed \( v \) and the inside area of the tube is \( A \).

In terms of the given quantities, derive an expression for the angular acceleration of the sprinkler at the instant when it starts to rotate. Assume that at that instant the moment of inertia of the sprinkler (including water) about the axis is \( I \) and that water has a mass per unit volume of \( \rho \). The nozzle of the sprinkler is a perpendicular distance \( l \) from the axis as shown.

**View from top:**

\[ F \]

vertical axis of sprinkler about which rotation occurs and about which moment of inertia is \( I \)

\[ \tau = I \alpha \quad \therefore \alpha = \frac{\tau}{I} \]

\( \tau = 2 FL \) where \( F = \) force due to ejected water as indicated.

By 3rd law \[ F = \frac{dP}{dt} = v \frac{dm}{dt} = v \cdot \rho A \]

\[ \therefore \alpha = \frac{2 v^2 A \rho L}{I} \]
\( F(t) \) (Vibrations)

- **FBD**:
  \[
  \begin{array}{c}
  m_1 \quad \text{assumes spring in tension} \\
  k_1 (x_1 - x_2) \quad \text{and} \quad k_2 x_2 \\
  b_1 \dot{x}_1 \quad \text{and} \quad b_2 \dot{x}_2
  \end{array}
  \]

**Newton's Law**:
\[
\begin{align*}
F - b_1 \dot{x}_1 - k_1 (x_1 - x_2) &= m_1 \ddot{x}_1 \\
k_1 (x_1 - x_2) - k_2 x_2 - (b + b_2) \dot{x}_2 &= m_2 \ddot{x}_2
\end{align*}
\]

Re-arranging:
\[
\begin{align*}
m_1 \ddot{x}_1 + b_1 \dot{x}_1 + k_1 x_1 &= F + k_1 x_2 \\
m_2 \ddot{x}_2 + (b + b_2) \dot{x}_2 + (k_1 + k_2) x_2 &= k_1 x_1
\end{align*}
\]

Take Laplace transforms:
\[
\begin{align*}
(m_1 s^2 + b_1 s + k_1) X_1(s) &= F(s) + k_1 X_2(s) \\
(m_2 s^2 + (b + b_2) s + k_1 + k_2) X_2(s) &= k_1 X_1(s)
\end{align*}
\]

Solve in the Laplace domain:
\[
\begin{align*}
(m_2 s^2 + (b + b_2) s + k_1 + k_2) X_2(s) &= k_1 \left( \frac{F(s) + k_1 X_2(s)}{m_1 s^2 + b_1 s + k_1} \right)
\end{align*}
\]

\[
X_2(s) \left[ \frac{m_2 s^2 + (b + b_2) s + k_1 + k_2}{m_1 s^2 + b_1 s + k_1} \right] = k_1 F(s) + k_1^2 X_2(s)
\]
\[
X_2(s) = \frac{k_1 F(s)}{(m_2 s^2 + (b_1 + b_2) s + k_1 + k_2)(m_1 s^2 + b_1 s + k_1) - \kappa_1^2}
\]
\[
X_1(s) = \frac{F(s)}{(m_2 s^2 + (b_1 + b_2) s + k_1 + k_2)(m_1 s^2 + b_1 s + k_1) - \kappa_1^2}
\]

Frequencies and damping calculations can be done on the basis of the denominator. Attempt a factorization of the form

\[
(s^2 + 2 \xi_1 \omega_1 s + \omega_1^2)(s^2 + 2 \xi_2 \omega_2 s + \omega_2^2)
\]

\[
= s^4 + (2 \xi_1 \omega_1 + 2 \xi_2 \omega_2) s^3 + (\omega_2^2 + \omega_1^2 + 4 \xi_1 \xi_2 \omega_1 \omega_2) s^2 + (2 \xi_1 \omega_1 \omega_2^2 + 2 \xi_2 \omega_2 \omega_1^2) s + \omega_1^2 \omega_2^2
\]

While the denominator can be expanded as

\[
s^4 + \left[\frac{m_1 (b_1 + b_2) + b_1 m_2}{m_1 m_2}\right] s^3 + \left[\frac{m_1 (k_1 + k_2) + m_2 k_1 + b_1 (b_1 + b_2)}{m_1 m_2}\right] s^2
\]

\[
+ \left[\frac{(b_1 + b_2) k_1 + b_1 (k_1 + k_2)}{m_1 m_2}\right] s + \kappa_1 \kappa_2
\]

(the \(m_1 m_2\) factor has been extracted to make it monic)
Substitute into (3) and (2):

\[
\begin{cases}
\frac{1}{w_2^2} + w_2^2 + 4y(0.0527-y) = 3.5001 \\
w_2^2(0.0527-y) + y/w_2^2 = 0.027
\end{cases}
\]

Change variables: \( z = w_2^2 \) (\( z \) needs to be positive)

\[
\begin{cases}
\frac{1}{z} + z + 4y(0.0527-y) = 3.5001 \\
z(0.0527-y) + \frac{y}{z} = 0.027
\end{cases}
\]

Solve for \( y \) from the 1st eq:

\[
\frac{1}{z} + z - 3.5001 = 4y^2 - 0.2108y
\]

\[
4y^2 - 0.2108y + (3.5001 - \left(\frac{1}{z} + z\right)) = 0
\]

\[
y = \frac{0.2108 \pm \sqrt{0.2108^2 - 16(3.5001 - \frac{1}{z} - z)}}{8}
\]

Substitute into the second eq:

\[
z\left(0.0527 - \frac{0.2108 \pm \sqrt{(0)}}{8}\right) + \frac{0.2108 \pm \sqrt{(0)}}{8z} = 0.02
\]
Let
\[ f(z) = \frac{\pi}{2} \left( 0.0527 - \frac{0.2108 \pm \sqrt{\cdot}}{8} \right) + \frac{0.2108 \pm \sqrt{\cdot}}{8z} - 0.02 \]

\[ f(z) = \left( \frac{0.2108 \pm \sqrt{16(z^2 + z) - 55.9572}}{8} \right) \left[ \frac{1}{z} \frac{z}{z} \right] + 0.5227 z - 0.025 \]

For a solution to \( f(z) = 0 \) to exist we need

\[ 16(z^2 + z) \geq 55.9572 \]

\[ (1 + z^2) \geq 3.4973 z \]

\[ z^2 - 3.4973z + 1 \geq 0 \]

\[ (z - 3.1832)(z - 0.3142) \geq 0 \]

\[ \Rightarrow z \geq 3.1832 \quad \text{(use to guide guessing)} \]

Plot a few values:

- Solution will lie between 3.1832 and 0.3142 using the "+"
Using numerical values:

\[ S^4 + 0.1055 S^3 + 3.5001 S^2 + 0.0545 S + 1 = 0 \]

Matching coefficients:

\[
\begin{align*}
2(\xi_1 \omega_1 + \xi_2 \omega_2) &= 0.1055 \\
\omega_1^2 + \omega_2^2 + 4 \xi_1 \omega_1 \xi_2 \omega_2 &= 3.5001 \\
2(\xi_1 \omega_1^2 + \xi_2 \omega_2^2) &= 0.054 \\
\omega_1^2 \omega_2^2 &= 1 &\Rightarrow& \omega_1 \omega_2 = 1
\end{align*}
\]

Change the variables:

\[ \xi_1 \omega_1 = x \]
\[ \xi_2 \omega_2 = y \]

Re-write:

\[
\begin{align*}
x + y &= 0.0527 \quad \text{(1)} \\
\omega_1^2 + \omega_2^2 + 4xy &= 3.5001 \quad \text{(2)} \\
x \omega_2^2 + y \omega_1^2 &= 0.027 \quad \text{(3)} \\
\omega_1 \omega_2 &= 1 \quad \text{(4)}
\end{align*}
\]

4 eqs, 4 unknowns.

From (1):

\[ x = 0.0527 - y \]

From (4):

\[ \omega_1 = \frac{1}{\omega_2} \]
Heretofore a few times by bisection:

\[ z \approx 3.186 \]

\[ w_2 = 1.785 \quad \rightarrow \quad w_1 = 0.5602 \]

\[ y = 0.0516 \quad \rightarrow \quad x = 0.0017 \quad \rightarrow \quad \xi_1 = 0.00207 \quad \xi_2 = 0.0029 \]

Then values can be inaccurate.

Since \( \xi_1, \xi_2 \) are betw. 0 and 1, the system is underdamped.

Note: Solution has been shown without using special formulas or advanced solvers.

Matrix methods can simplify solution.
Heat Transfer

\[ h_i = \frac{1000 \text{ W}}{m^2\text{K}} \]

\[ h_o = \frac{2\pi \text{ W}}{m^2\text{K}} \]

\[ k = \frac{56.5 \text{ W}}{m\text{K}} \]

**Inner Convection:**
\[ Q = h_i \pi \phi_i L (500 - T_i) \]

**Conduction:**
\[ Q = \frac{2\pi k L}{\ln (\phi_o/\phi_i)} (T_o - T_i) \]

**Outer Convection:**
\[ Q = h_o \pi \phi_o L (T_o - 25) \]

\[ \frac{Q}{L} = h_o \pi \phi_o (500 - T_i) = b(500 - T_i) \]

\[ \frac{Q}{L} = \frac{2\pi k}{\ln (75/60)} = a(T_i - T_o) \]

\[ \frac{Q}{L} = h_o \pi \phi_o (T_o - 25) = c(T_o - 25) \]

**Solution:**
\[ T_i = \frac{500b(1 + \frac{c}{a}) + 25c}{b(1 + \frac{c}{a}) + c} \]
Numerically

\[ a = \frac{2\pi \kappa}{\ln(T_f/T_0)} = 1590.904 \] (meters)

\[ b = \hbar \pi \phi_i = 188.4956 \]

\[ c = \hbar_0 \pi \phi_0 = 4.7124 \]

\[ T_i = 483.45 \degree C \]

\[ T_0 = 487.08 \degree C \]

\[ q = \frac{\Theta}{L} = 2.1775 \times 10^3 \frac{W}{m} \]
Duration: 4 hours — Open books and notes.
WORK ON 4 QUESTIONS
The use of computers, cameras, cell phones and other communication devices is prohibited.

1. (Thermodynamics) A Rankine cycle operates with water between 20 bar (only saturated liquid at pump outlet) and 0.07 bar. Considering isentropic processes for the pumping and expansion, calculate:

   1. The quality at the exit of the turbine
   2. The work required by the pump in kJ/kg (assume approximately constant volume process)
   3. The work done by the turbine in kJ/kg

2 (Control Systems) Sketch the root locus, finding asymptote centers and angles.

   \[ G(s) = \frac{s^2 + 0.5s + 4}{(s - 1)^2(s^2 + 10s + 100)(s + 2)} \]

Tell if a value of gain can exist that stabilizes the system

3. (Solid Mechanics) A cylindrical tank with a diameter of 30" was built by rolling 3/8" steel plate along a helix forming 25° with the horizontal, as shown in the figure. The tank is pressurized to a gage pressure of 180 psi. Determine the stresses along the weld and perpendicular to the weld.
4 (Fluid Mechanics) Water at 221 kPa enters a 90° reducing elbow with an inlet area of 0.01 m². The area of the outlet is 0.0025 m² and the exit velocity is 16 m/s, at atmospheric pressure. Calculate the system of forces required to keep the elbow static.

5 (Vibrations) Find expressions for the natural frequencies and damping ratios of the system shown in the figure.

5 (Heat Transfer) An aluminum plate, heated to a uniform temperature of 227°C, is allowed to cool while vertically suspended in a room where the ambient air and surroundings are at 27°C. The plate is 0.3 m square with a thickness of 15 mm and an emissivity of 0.25.

1. Develop an expression for the time rate of change of the plate temperature, assuming the temperature to be uniform at any time.

2. Determine the initial rate of cooling (K/s) when the plate temperature is 227°C.

Use the following properties: density=2770 kg/m³; conductivity=186 W/m°C; heat capacity=983 J/kg°C K.
DRE Engineering Exam. — Spring 2009

Open-book portion. * Solution *

1) Thermodynamics

Since no information is provided regarding superheating at point 3, assume only saturated vapor enters the turbine.

Saturation temperatures: From tables: \( T_3 = 212.4 \, ^\circ\text{C} \)
\( @ \) 20 bar

\( T_1 = T_4 = 40 \, ^\circ\text{C} \)
\( @ \) 0.07 bar

Enthalpies: \( h_3 = 2799.5 \, \text{kJ/kg} \) (table)

Entropies: \( s_3 = 6.34 \, \text{kJ/kg}^\circ\text{K} \) (table)

\( s_{g1} = 8.25 \, \text{kJ/kg}^\circ\text{K} \), \( s_{f1} = 0.57 \, \text{kJ/kg}^\circ\text{K} \)
1) Quality at 4: \[ x_4 = \frac{s_3 - s_{A1}}{s_8 - s_{f1}} = 0.75 \]

Then the enthalpy \( h_4 \) can be found to be:

\[ h_4 = \frac{1678.5 + 0.75(2406.7)}{h_{f1}} x_4 \frac{h_{fg1}}{h_{f1}} = 1972.5 \text{ kJ/kg} \]

2) The work required by the pump:

\[ \omega_b = - \int_{p_1}^{p_2} V \, dp = -V_1 \int_{p_1}^{p_2} dp = -V_1 (p_2 - p_1) \]

because of constant volume assumption

\[ \omega_b \approx -2 \text{ kW/kg} \]

3) The work done by the turbine:

\[ \omega_t = h_3 - h_4 = 827 \text{ kJ/kg} \]
2) Control Systems:

Zeros: \[ s = -0.25 \pm 1.9843z \]

Poles: \[ s = 1, \: s = 1 \]
\[ s = -5 \pm 8.6603z \]
\[ s = -2 \]

\( n = 5 \) poles, \( m = 2 \) zeros. \( \Rightarrow \) 3 poles \( s \to \infty \)

Asymptote center: \[ \sigma_A = \frac{\Sigma p - \Sigma z}{n-m} = 3.166 \]

Angles of asymptotes: \[ \phi_A = 180\left(\frac{2k+1}{n-m}\right), \: k=0,1,...,n-m-1 \]

\( \{60^\circ, 180^\circ, 300^\circ\} \)

A value of gain may exist that stabilizes the system.
We would need to determine if \( k_1 > k_2 \).
This tank is clearly "thinned". We use appropriate formulas for the principal stresses.

Tangential: \( \sigma_t = \frac{Pr}{t} = \frac{150 \times 15}{3/8} \)
Longitudinal: \( \sigma_L = \frac{1}{2} \sigma_t = 3600 \) psi

\( \sigma_t = 7200 \) psi
\( \sigma_L = 3600 \) psi

Now we need to rotate the stresses to line up with the weld: (Note that \( \sigma_t \) and \( \sigma_L \) are principal stresses)

\[ R = \frac{\sigma_t - \sigma_L}{2} = 1800 \text{ psi} \]

\[ \sigma_W = \frac{\sigma_t + \sigma_L}{2} - R \cos 50^\circ \]

\[ \sigma_W = 4240 \text{ psi} \] (+)

\[ \tau_W = R \sin 50^\circ = 1379 \text{ psi} \]
Conservation of momentum in X:

\[
\mathbf{F}_{5x} = \int \mathbf{u} \rho \mathbf{v} \cdot d\mathbf{A} = \int_{A_1} \mathbf{u} \rho \mathbf{v} \cdot d\mathbf{A}
\]

\[
P_1 A_1 + \rho A_3 - \rho (A_3 + A_7) + F_x = \int_{A_7} \mathbf{u} \rho \mathbf{v} \cdot d\mathbf{A}
\]

\[
(p_1 - p_0) A_1 + F_x = -\int_{A_1} \mathbf{u} / \rho \mathbf{v} \cdot d\mathbf{A}
\]

\[
P_x = -p_1 A_1 - u_1 / \rho v_1 A_1
\]

We need to determine \( u_1 \):

By continuity: \( u_1 = u_2 \frac{A_2}{A_1} = 46 \text{ m/s} \)

\( \implies P_x = -1.36 \text{ kN} \)

For the \( y \) direction:

\[
\rho A_4 + \rho A_2 - \rho A_2 - \rho A_2 + F_{By} + F_y = \int_{A_2} \mathbf{v} / \rho \mathbf{v} \cdot d\mathbf{A}
\]

\[
F_{By} + F_y = u_2 / \rho v_2 A_2
\]

\[
F_y = -F_{By} + u_2 / \rho v_2 A_2
\]
neglecting $F_y$ we get $F_y = -635\text{N}$
5) ( Vibrations )

Free-body diagram:

\[
\begin{array}{c}
\text{F} \\
\downarrow
\end{array}
\begin{array}{c}
\text{m}_1 \\
\rightarrow
\end{array}
\begin{array}{c}
k_1 (x_1 - x_2) \\
\rightarrow
\end{array}
\begin{array}{c}
x_1 \\
\leftarrow
\end{array}
\begin{array}{c}
\text{m}_2 \\
\rightarrow
\end{array}
\begin{array}{c}
k_2 x_2 \\
\rightarrow
\end{array}
\begin{array}{c}
x_2 \\
\leftarrow
\end{array}
\begin{array}{c}
\text{F}_0
\end{array}
\]

Newton's Laws:

\[F - k_1 (x_1 - x_2) = m_1 \ddot{x}_1\]
\[k_1 (x_1 - x_2) - k_2 x_2 - b \dot{x}_2 = m_2 \ddot{x}_2\]

Taking the Laplace transform:

\[
\begin{cases}
\left[ m_1 s^2 + k_1 \right] X_1 (s) = F(s) + k_1 X_2 (s) \\
\left[ m_2 s^2 + (k_1 + k_2) + bs \right] X_2 (s) = k_1 X_1 (s)
\end{cases}
\]

Solve for \( X_1 (s) \) and \( X_2 (s) \):

\[X_1 (s) = \frac{1}{k_1} \left( m_2 s^2 + k_1 k_2 + bs \right) X_2 (s)\]
\[
\Rightarrow \left[ \frac{m_2 s^2 + k_1 k_2 + bs}{k_1} \right] X_2 (s) = F(s) + k_1 X_2 (s)
\]

\[
\frac{m_2 s^2 + bs + k_1 k_2}{k_1} \left( m_1 s^2 + k_1 \right) - k_1 \left[ \frac{(m_2 s^2 + bs + k_1 k_2)}{k_1} \right] X_2 (s) = F(s)
\]

\[X_2 (s) = \frac{F(s)}{\left( \frac{m_2 s^2 + bs + k_1 k_2}{k_1} \right) \left( m_1 s^2 + k_1 \right) - k_1}\]
\[ X_1(s) = \frac{F(s)}{k_1 \left[ \frac{m_2 s^2 + bs + k_1 + k_2}{k_1} \right] \left( m_1 s^2 + k_1 \right) - k_1} \]

The characteristic equation

\[ \frac{1}{k_1} \left( m_2 s^2 + bs + k_1 + k_2 \right) \left( m_1 s^2 + k_1 \right) - k_1 = 0 \]

is a 4th order polynomial which (assuming underdamped case) leads to complex solutions of the form

\[ \left\{ - \xi_1 \omega_1 \pm \omega_1 \sqrt{1 - \xi_1^2} \chi \\
- \xi_2 \omega_2 \pm \omega_2 \sqrt{1 - \xi_2^2} \chi \right\} \]

The frequencies and damping ratios are obtained directly.
S.19 An aluminum plate, heated to a uniform temperature of 227° C, is allowed to cool while vertically suspended in a room where the ambient air and surroundings are at 27° C. The plate is 0.3 m square with a thickness of 15 mm and an emissivity of 0.25.

a) Develop an expression for the time rate of change of the plate temperature, assuming the temperature to be uniform at any time.

b) Determine the initial rate of cooling (K/s) when the plate temperature is 227° C.

c) Justify the uniform plate temperature assumption.

**KNOWN:** Aluminum plate (alloy 2024) at uniform temperature 227° C suspended in a room where the ambient air and surroundings are at 27° C.

**FIND:** (a) Expression for time rate of change of the plate, (b) Initial rate of cooling (K/s) when plate temperature is 227° C, (c) Justify uniform plate temperature assumption.

**SCHEMATIC:**

**ASSUMPTIONS:** (1) Plate temperature is uniform, (2) Ambient air is quiescent and extensive, (3) Surroundings are large compared to plate.

**PROPERTIES:** Table A-1, Aluminum alloy 2024 (T = 500K): \( \rho = 2770 \text{ kg/m}^3 \), \( k = 188 \text{ W/m·K} \), \( c = 683 \text{ J/kg·K} \); Table A-4, Air (T\text{f} = 400K, 1 atm): \( \nu = 26.41 \times 10^{-6} \text{ m}^2/\text{s} \), \( k = 0.0338 \text{ W/m·K} \), \( \alpha = 38.3 \times 10^{-6} \text{ m}^2/\text{s} \), Pr = 0.690.

**ANALYSIS:** (a) From an energy balance on the plate considering free convection and radiation exchange, \(-\dot{E}_{\text{out}} = \dot{E}_{\text{st}}\),

\[
-\varepsilon_0 c_0 A_c (T_i - T_w) - c_2 A_c \varepsilon(T_i - T_w)^4 = \rho A c \frac{dT}{dt} + \frac{dT}{dt} = -\varepsilon c \frac{h_i(T_i - T_w)}{T_i} + \varepsilon \varepsilon (T_i - T_w)
\]

where \( T_i \) is the plate temperature assumed to be uniform at any time.

(b) To evaluate \( (dT/dt) \), estimate \( \bar{h}_L \). Find first the Rayleigh number, \( Ra_L = \varepsilon \varepsilon (T_i - T_w) \times \frac{9.83 \text{ m/s}^2 (1/400K) (227 - 27)K \times (0.3m)\varepsilon}{28.41 \times 10^{-6} \text{ m}^2/\text{s} \times 38.3 \times 10^{-6} \text{ m}^2/\text{s}} = 1.309 \times 10^8 \).

Eq. 9.27 is appropriate; substituting numerical values, find

\[
\bar{N}_U_L = 0.68 + \frac{0.670Ra_L^{1/4}}{[1 + (0.492/Pr)^{9/16}]^{1/9}} = 0.68 + \frac{0.670(1.308 \times 10^8)^{1/4}}{[1 + (0.492/0.690)^{9/16}]^{1/9}} = 55.5
\]

\[
\bar{h}_L = \bar{N}_U_L k/L = 55.5 \times 0.0338 \text{ W/m·K}/0.3m = 6.25 \text{ W/m}^2\cdot\text{K}
\]

\[
\frac{dT}{dt} = \frac{-2}{2770 \text{ kg/m}^3 \times 0.015m \times 983 \text{ J/kg·K}} 
\]

\[
[0.25W/m^2·K(227 - 27)K + 0.25(5.67 \times 10^{-8}W/m^2·K)(500^4 - 300^4)K] = -0.099K/s. < 1
\]

(c) The uniform temperature assumption is justified if the Biot number criterion is satisfied. With \( L_e \equiv (V/A_c) = (c_2 \varepsilon c_0 A_c) = \varepsilon \) and \( \bar{h}_\text{rad} = \bar{h}_\text{conve} \), Bi = \( \bar{h}_\text{rad} \varepsilon /k \leq 0.1 \).

Using the linearized radiation coefficient relation, find

\[
\bar{h}_\text{conve} = c_0(T_i + T_w)/(T_i + T_w) = 0.25(5.67 \times 10^{-8}W/m^2·K)^4 \times (500^4 + 300^4)K^2 = 3.88 \text{ W/m}^2·\text{K}
\]

Hence, Bi = (6.25 + 3.86)W/m^2·K(0.015m)/186 W/m·K = 8.15x10^{-4}. Since Bi << 0.1, the assumption is appropriate.