Mathematics for Control Theory

Elements of Analysis p.2
Normed and Inner Product Spaces
Banach and Hilbert Spaces

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Reading materials

We will use:

Normed Linear Spaces

Let $X$ be a Linear vector space over some field $\mathbb{F}$. A mapping from $X$ into $\mathbb{R}$ is called a norm on $X$ (denoted $||x||$ for all elements $x \in X$) if $\forall x, y \in X$, $\forall \alpha \in \mathbb{F}$:

1. $||x|| \geq 0$
2. $||x|| = 0 \iff x = 0$
3. $||\alpha x|| = |\alpha||x||$
4. $||x + y|| \leq ||x|| + ||y||$

The pair $\{X; ||.||\}$ is called a normed linear space. Note that 4. implies

$$\left|\sum_{i=1}^{n} x_i \right| \leq \sum_{i=1}^{n} |x_i|$$
Normed Linear Spaces Induce Metric Spaces

Let \( \{X; \| \cdot \| \} \) be a normed linear space. Define a function \( \rho : X \times X \mapsto \mathbb{R} \) by

\[
\rho(x, y) = \|x - y\|
\]

for all \( x, y \in X \). Then Theorem 6.1.2 in Michel and Herget indicates that \( \{X; \rho\} \) is a metric space.

Prove this as an exercise.

Observe the following:

1. We can interpret a normed linear space as a metric space with metric
   \[
   \rho(x, y) = \|x - y\|.
   \]

2. Therefore, all properties that hold for metric spaces also hold for normed linear spaces.

3. Only an adjustment to the terminology is necessary. For example, an open ball with center \( x_0 \) and radius \( r \) in a normed linear space is defined by:
   \[
   S(x_0, r) = \{ x \in X : \|x - x_0\| < r \}
   \]
Banach Spaces

A complete normed linear space is called a Banach space. That is, \( \{X; ||.||\} \) is defined to be a complete normed linear space iff \( \{X; \rho\} \) is a complete metric space, with \( \rho(x, y) = ||x - y|| \).

Examples:

1. Consider \( X = \mathbb{C}^n \). For \( x = (\zeta_1, \zeta_2, ... \zeta_n) \in X \) define the \( p \)-norm as

\[
||x||_p = \left( \sum_{i=1}^{n} |\zeta_i|^p \right)^{1/p}
\]

with \( 1 \leq p < \infty \).

The corresponding \( p \)-metric was studied earlier, and we saw that it defined a complete metric space. Therefore \( \{\mathbb{C}^n; ||.||_p\} \) is complete.
Examples...

2. Define

\[ \|x\|_\infty = \max_i |\zeta_i| \]

The corresponding metric was also studied earlier and it defines a complete metric space. Therefore \( \{\mathbb{C}^n; \|\cdot\|_\infty\} \) is complete.

3. \( l_p \) spaces are frequently found in the analysis of discrete-time control systems. It’s important to use a lower-case \( l \). Denote the set of infinitely-long vectors (sequences) of complex (or real in particular) numbers as \( X = \mathbb{C}^\infty \). The set \( l_p \) is a linear subspace of \( X \) defined by:

\[ l_p = \{x \in X : \sum_{i=1}^{n} |\zeta_i|^p < \infty\} \]

The \( l_p \) norm of any sequence of \( l_p \) is defined by

\[ \|x\|_p = \left( \sum_{i=1}^{n} |\zeta_i|^p \right)^{1/p} \]
Examples...

4. \( \{ l_p; \| \cdot \|_p \} \) are also Banach spaces for \( 1 \leq p < \infty \).

5. Also define

\[
l_\infty = \{ x \in X : \sup_i |\zeta_i| < \infty \}
\]

and

\[
\| x \|_\infty = \sup_i \{ |\zeta_i| \}
\]

Again, \( \{ l_\infty; \| \cdot \|_\infty \} \) is a Banach space.
Completeness/Incompleteness of certain function spaces

Consider $X = C[a, b]$. We know $X$ is a linear vector space. Define the norms

$$||x||_p = \left[ \int_a^b |x(t)|^p dt \right]^{1/p}$$

for all $x \in X$ and $1 \leq p < \infty$.

Although it can be verified that $\{C[a, b]; ||\cdot||_p\}$ are normed linear spaces, we saw that the corresponding metric spaces are not complete, so these normed spaces are not Banach spaces.

However, taking

$$||x||_\infty = \sup_{t \in [a, b]} |x(t)|$$

does correspond to a complete metric space. Hence, $\{C[a, b]; ||\cdot||_\infty\}$ is a Banach space.
Product Normed Spaces

Paralleling the discussion of product metric spaces, suppose \( \{X; ||.||_x\} \) and \( \{Y; ||.||_y\} \) are normed spaces. How do we define a norm for \( X \times Y \)?

First, \( X \times Y \) must be a linear space. This can be accomplished with the following addition and multiplication rules:

\[
(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)
\]

\[
\alpha(x, y) = (\alpha x, \alpha y)
\]

Then a valid norm for \( X \times Y \) is simply

\[
||(x, y)|| = ||x||_x + ||y||_y
\]

If the individual spaces are Banach, so will be their product under this norm.
Some Convexity Results

Let $X$ be a real normed linear space. Given $x, y \in X$, define the segment joining $x$ and $y$ by the set:

$$xy = \{z \in X : z = \alpha x + (1 - \alpha)y \ 0 \leq \alpha \leq 1\}$$

Let $Y \subset X$. We say that $Y$ is convex if $xy \subset Y \ \forall x, y \in Y$.

1. Let $S$ be any family of convex sets. Then $\bigcap_{Y \in S} Y$ is also convex.
2. Definition: The convex hull of $Y$ is the intersection of all convex sets which contain $Y$. It is the smallest convex set containing $Y$.
3. The closure of $Y$ is convex whenever $Y$ is convex.
Continuity of norm and bounded functionals

The norm is a function from $X$ into $\mathbb{R}$. Theorem 6.1.15 shows that this function is continuous on $X$.

Recall that a linear functional $f$ is a mapping from a linear space to the reals:

$$ f : X \mapsto \mathbb{R} $$

A linear functional must satisfy superposition and homogeneity, as seen in the previous handout. A functional $f$ is bounded if $\exists M \geq 0$ s.t.

$$ |f(x)| \leq M ||x|| $$

for all $x \in X$. For linear functionals, continuity and boundedness are the same thing, as seen in Th. 6.5.5.

Next, we define a norm for bounded linear functionals.
A norm for bounded functionals

Let $X^f$ be the linear space of all linear functionals on $X$ and let $X^*$ be the subset of $X^f$ formed bounded functionals. Define a mapping from $X^*$ into $\mathbb{R}$ by:

$$
\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}
$$

for all $f \in X^*$. Then (Th. 6.5.6):

1. $X^*$ is a linear subspace of $X^f$
2. $\|\cdot\|$ is a norm on $X^*$
3. $\{X^*; \|\cdot\|\}$ is a Banach space

Note that $X^*$ is Banach even if $X$ isn’t. Space $X^*$ is called the normed dual of $X$, or just the dual of $X$. 
Example

Take $X = \mathbb{R}^n$. Let $a \in X$ be fixed and define the functional $f : X \mapsto \mathbb{R}$ by the rule $f(x) = a^T x$ for all $x \in \mathbb{R}^n$.

1. Show that $f$ is a bounded linear functional
2. Find a formula for $\|f\|$ using the of the previous slide.
Alternative functional norm formulas

See Th. 6.5.10: Let $f$ be a bounded linear functional and $\|f\|$ its norm. Then

$$\|f\| = \inf_M \{ M : |f(x)| \leq M\|x\| \ \forall x \in X \}$$

$$\|f\| = \sup_{\|x\| \leq 1} \{|f(x)|\}$$

$$\|f\| = \sup_{\|x\| = 1} \{|f(x)|\}$$

Examine the norms for functionals on $C[a, b]$ in Examples 6.5.12-13 carefully.

Skip to Section 6.11.
Signal Norms of Interest in Control

In control systems analysis, it is important to establish the “size” of various signals. For instance, if we restrict the class of signals to continuous functions defined on finite intervals, we saw that

$$
\|x\|_p = \left[ \int_a^b |x(t)|^p dt \right]^{1/p}
$$

is a valid norm. A few limitations so far:

1. An infinite value of the norm could arise if we extend $b$ to infinity. But there is a subclass of functions for which the integral is still finite.

2. The normed space \( \{C[a, b]; \| \cdot \|_p \} \) is not complete for \( 1 \leq p < \infty \). Cauchy sequences where every element is a continuous function could fail to converge to a continuous function.

3. Signals found in engineering systems must include discontinuous functions.

These limitations motivate extension to a wider class of functions. The set of Lebesgue measurable functions will be outlined later in the course.
Inner Product Spaces

Let $X$ be a linear space over $\mathbb{C}$. An inner product on $X$ is a function $\langle ., . \rangle : X \times X \to \mathbb{C}$ satisfying:

1. $\langle x, x \rangle > 0$ for all $x \neq 0$ and $\langle x, x \rangle = 0$ if $x = 0$ (positive-definite function).
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in X$
3. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and all $\alpha, \beta \in \mathbb{C}$ (bilinearity)
4. $\langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle$ for all $x, y, z \in X$ and all $\alpha, \beta \in \mathbb{C}$

The combination $\{X; \langle ., . \rangle\}$ is called an inner product space.
Inner Product Spaces Induce Normed Spaces

Notational simplification: refer to the inner product space \( \{X; \langle ., . \rangle \} \) in context as just \( X \).

Let \( X \) be an inner product space. Then the function \( \| . \| : X \rightarrow \mathbb{R} \) defined by

\[
\| x \| = \langle x, x \rangle^{1/2}
\]

is a norm (satisfies the 4 conditions presented earlier). Also, the Cauchy-Schwarz inequality holds:

\[
|\langle x, y \rangle| \leq \| x \| \| y \|
\]

We can associate a normed linear space to every inner product space by using the above norm. As done before, we can in turn associate a metric space. All properties that hold for a metric spaces will hold for inner product spaces.

A complete inner product space is called a Hilbert space. Inner product spaces including incomplete ones are called pre-Hilbert spaces.
Important Hilbert Spaces

- \( \mathbb{C}^n \) with any suitable inner product is a Hilbert space. Note: Let \( x, y \in \mathbb{C} \) be represented as column vectors with \( n \) components. The proper way to compute their inner product is \( \bar{x}^T y \) and not just \( x^T y \). In Matlab: \( x’*y \) will conjugate as well as transpose \( x \). If for whatever reason you want to block the conjugation, use \( x.’*y \).

- The space of summable sequences \( l_p \) studied earlier: Let \( x = (\zeta_1, \zeta_2, ..) \) and \( y = (\eta_1, \eta_2, ...) \) be 2 sequences in \( l_p \). Define their inner product as

\[
\langle x, y \rangle = \sum_{i=1}^{\infty} \zeta_i \bar{\eta}_i
\]

We show in class that \( l_p \) is an inner product space for \( p = 2 \) only. Also, \( l_2 \) is a Hilbert space.

- While a discussion of Lebesgue measurable functions is pending, the space of definition is denoted \( L_p[a, b] \) and the inner product is defined as

\[
\langle f, g \rangle = \int_a^b f(t)g(t)dt
\]
Product Hilbert Spaces

Let \( \{X_i\} \) be a collection of Hilbert spaces over \( \mathbb{C} \), each with an inner product \( \langle ., . \rangle_i \) for \( i = 1, 2...n \). We can create a linear space with \( X = X_1 \times X_2 \times ...X_n \) with the usual component-wise addition between vectors and multiplication between vectors and scalars. How do we define an inner product?

The summation rule works again (we did this for metric and normed spaces):

\[
\langle x, y \rangle = \sum_{i=1}^{n} \langle x_i, y_i \rangle_i
\]

for all \( x = (x_1, x_2, ...x_n) \in X \) and for all \( y = (y_1, y_2, ...y_n) \in X \)

It can be verified that \( X \) is a Hilbert space.
Orthogonality in Inner-Product Spaces

Let $X$ be an inner product space. Two elements $x, y \in X$ are orthogonal (denoted $x \perp y$) if

$$\langle x, y \rangle = 0$$

A collection $\{x_\alpha : \alpha \in \mathcal{I}\}$ where $\mathcal{I}$ is an arbitrary index set is an orthogonal set if $x_\alpha \perp x_\beta$ for all $\alpha, \beta \in \mathcal{I}$. If in addition $||x_\alpha|| = 1$ for all $\alpha \in \mathcal{I}$, the set is orthonormal. Example 6.11.20 shows that the set of continuous complex functions:

$$f_n(t) = e^{2\pi i n t}, \quad n = 0, \pm 1, \pm 2...$$

is orthonormal.
Application to linear independence: The Gram matrix

Recall that the definition of linear independence is valid for any linear vector space, in particular for Hilbert spaces.

Th. 6.15.3: Let \( \{y_1, \ldots, y_n\} \) be elements of a Hilbert space \( X \). The set is linearly independent iff

\[
\det(G(y_1, y_2, \ldots, y_n)) \neq 0
\]

where \( G \) is the Gram matrix, whose elements are \( G_{ij} = \langle y_i, y_j \rangle \) for \( i, j = 1, 2, \ldots, n \).

Example: We looked at the linear independence/dependence of \( \{\sin t, \cos t\} \) and \( \{m_1x + b_1, m_2x + b_2\} \) from the definition and by using the Wronskian.

Knowing that these functions are elements of the Hilbert space \( L_2[a, b] \), verify our earlier results with the Gram matrix method.