Mathematics for Control Theory

Geometric Concepts in Control
Lie algebra of vector fields
Flows, Lie Brackets and Commutativity

Hanz Richter
Mechanical Engineering Department
Cleveland State University
Reading materials

Reference:


Directional Derivatives and Tangent Vectors

Let $f$ be a differentiable function $f : \mathcal{M} \rightarrow \mathbb{R}$. Given a point $a \in \mathcal{M}$, the directional derivative of $f$ at $a$ in the direction $v$ is

$$D_v f(a) = (\nabla f)(a)v$$

When $v \in T_a \mathcal{M}$, we have an equivalence class of curves where each curve $\gamma$ satisfies

$$\gamma'(0) = v$$

In this case, the directional derivative is

$$D_v f(a) = (f \circ \gamma)'(0)$$

In the above, we can use any $v \in T_a \mathcal{M}$. In our next definition, the Lie derivative of a vector field, $v$ is the image of the vector field at $a$. 
Lie derivative of a scalar function with respect to a vector field

A smooth vector field on the submanifold $\mathcal{M}$ is a $C^\infty$ mapping $f : \mathcal{M} \mapsto \mathbb{R}^n$ such that $f(x) \in T_x \mathcal{M}$ for all $x \in \mathcal{M}$.

From now on, we assume that vector fields are smooth. (This is used as a condition for existence of solutions to differential equations on manifolds. We comment on discontinuous vector fields and differential inclusions, as needed in variable structure controls (SMC) analysis). Also, higher-order derivatives will be calculated.

Let $f : \mathcal{M} \mapsto \mathbb{R}^n$ be a smooth vector field and let $h : \mathcal{M} \mapsto \mathbb{R}$ a smooth scalar function. The Lie derivative of $h$ with respect to $f$, denoted $L_fh$ is simply the directional derivative in the direction of $f$:

$$L_fh = \nabla hf$$
Lie derivatives as operators

The Lie derivative can be applied recursively:

\[ L_f(L_f h) = \nabla (L_f h) f = L_f^2 h \]

Also, we can use various vector fields:

\[ L_g(L_f h) = \nabla (L_f h) g = L_g L_f h \]

Do you expect commutativity?

For \( h(x, y) = x - y^2 \), \( g(x, y) = [x \ y]^T \) and \( f(x, y) = [y \ x^2]^T \), calculate \( L_g L_f h \) and \( L_f L_g h \)
The Lie Bracket

Let $f$ and $g$ be two smooth vector fields. The Lie bracket of $f$ and $g$ is another vector field defined by

$$[f, g] = \nabla g f - \nabla f g$$

The notation $\text{ad}_f g$ is also used for $[f, g]$ (given a fixed $f$, $\text{ad}_f g$ is the adjoint action of $f$ on the set of all smooth vector fields on the manifold).

The Lie bracket defines a non-associative algebra of smooth vector fields on a manifold. The algebraic properties are:

1. **Bilinearity**: The same property that is satisfied by inner products.
2. **Antisymmetry**: $[f, g] = -[g, f]$
3. **Jacobi identity**: $L_{[f,g]} h = L_{\text{ad}_f g} h = L_f L_g h - L_g L_f h$
Recursive Lie Brackets

\[ \text{ad}_{f^2g} = [f, \text{ad}_f g] \]

This can be worked out using the Jacobi identity:

\[ \text{ad}_{f^2g} h = L_{f^2} L_g h - 2L_f L_g L_f h + L_g L_{f^2} h \]

Exercise: Follow the proof of the Jacobi identity in Slotine and Li and use it to verify the above formula.
Suppose we have a differential equation on a manifold

\[ \dot{x} = f(x) \]

Recall that \( f \) assigns a vector \( f(x) \in T_x\mathcal{M} \), and a solution \( x(t) \) (called an integral curve) is such that for every \( t \), the tangent to the curve matches the value of the vector field there, \( f(x(t)) \).

The flow of the vector field at \( t \) is a mapping \( \Phi^t_f : \mathcal{M} \to \mathcal{M} \) that takes a point in \( \mathcal{M} \) (think of it as an initial condition) and returns the point on the integral curve that is reached at time \( t \).

Recall the semigroup property used as one of the axioms in the general definition of a dynamic system. In terms of flows:

\[ \Phi^t_f \circ \Phi^s_f = \Phi^{s+t}_f \]
Flows and Commutativity

Note that $\Phi_f^0$ is the identity mapping. Now, by the semigroup property:

$$\Phi^{-t}_f \circ \Phi^t_f = \Phi^0_f = \Phi^t_f \circ \Phi^{-t}_f$$

This means that the flow has a smooth inverse and constitutes a diffeomorphism.

Suppose there are two vector fields $f$ and $g$, each one generating its own flow. In general

$$\Phi^t_f \circ \Phi^t_g \neq \Phi^t_g \circ \Phi^t_f$$

Flows don’t have to commute in general. If the vector fields are constant, they will commute. The following is stated without proof:

$$\Phi^t_f \circ \Phi^t_g = \Phi^t_g \circ \Phi^t_f \iff [f, g] = 0$$
Commutativity...

A quick derivation shows that $\Phi_f^t \circ \Phi_g^t = \Phi_g^t \circ \Phi_f^t$ is equivalent to

$$\Phi_g^{-t} \circ \Phi_f^{-t} \circ \Phi_g^t \circ \Phi_f^t = \text{identity}$$

The above says: \textit{when the flows are commutative, we can follow $f$ forward in time, then follow $g$ forward, then follow $f$ backward and finally follow $g$ backward and we end in the same place.}

In turn, this is equivalent to $[f, g] = 0$. It is commonly said that the Lie bracket measures “lack of commutativity”.

Controllability will be studied in a future lecture, but for now we notice that a repeating sequence of backward and forward motions along two flows can be used to control the motion of $x$ in the manifold. We illustrate this with a classic example.
Lie brackets are a powerful tool for the analysis of mechanical control systems. Consider a dynamic model of a vehicle accounting for kinematics only:
Car Kinematics Model

We outline the approximations leading to the differential equation model below. These differential equations evolve on a 4-dimensional manifold described by configuration coordinates \((x_p, y_p, \phi, \theta)\). Notes:

■ All infinitesimal displacements of the two axles must remain perpendicular to them, indicating that there is no side slip.

■ Our controls are the angular velocity of the steering axle \((u_2 = \dot{\theta})\) and the linear speed perpendicular to the front axle \((u_1)\). (This velocity is proportional to tire angular speed when there’s no slippage)

\[
\begin{align*}
\dot{x}_p &= \cos(\phi + \theta)u_1 \\
\dot{y}_p &= \sin(\phi + \theta)u_1 \\
\dot{\phi} &= \frac{\sin(\theta)}{l}u_1 \\
\dot{\theta} &= u_2
\end{align*}
\]
Steer and Drive Vector Fields

For simplicity, take $l = 1$. Define the state vector as 
$x = [x_p, y_p, \phi, \theta]^T = [x_1, x_2, x_3, x_4]^T$. Then the vector field in the previous system can be viewed as a linear combination

$$
\dot{x} = d(x)u_1 + s(x)u_2
$$

where $d(x)$ represents driving and $s(x)$ steering:

$$
d(x) = \begin{bmatrix}
\cos(x_3 + x_4) \\
\sin(x_3 + x_4) \\
\sin(x_4) \\
0
\end{bmatrix} \\
s(x) = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
$$

We can switch each control between two values (say ±1) to choose which flows to follow and in which direction.
Steer, then drive

Let’s compute the bracket of $s$ and $d$

$$[\text{Steer, Drive}] = \begin{bmatrix} -\sin(x_3 + x_4) \\ \cos(x_3 + x_4) \\ \cos(x_4) \\ 0 \end{bmatrix}$$

The composition of flows indicates turning the wheels clockwise (without moving the car), followed by driving while holding the steering wheel fixed. We can call this “Wriggle”.

We note that the bracket is not zero, so if we perform a sequence of steers and drives following the pattern

$$\Phi_g^{-t} \circ \Phi_f^{-t} \circ \Phi_t^t \circ \Phi_g^t \circ \Phi_f^t$$

we don’t obtain the identity (the car moves in some direction).
Wiggle, then drive

Now follow Wriggle with another Drive. We obtain:

\[
\begin{bmatrix}
-\sin(x_3) \\
\cos(x_3) \\
0 \\
0
\end{bmatrix}
\]

The resulting vector field (call it Slide) points in the direction \((-\sin(\gamma), \cos(\gamma))\) which is perpendicular to the car! Then, it is possible to slide in/out the parking spot without any side slip on the wheels (that’s how we park/unpark in parallel).

We normally use: Wriggle, Drive, -Wriggle, -Drive, Wriggle, Drive....

Exercise: Verify that \([\text{Steer}, \text{Wriggle}]=\text{-Drive}\) and \([\text{Slide}, \text{Z}]=0\), for \(\text{Z} = \text{Steer}, \text{Drive}\) or \(\text{Wriggle}\). What does this mean?