ESC794: Special Topics: Model Predictive Control

Nonlinear MPC Analysis : Part 3
Reference: *Nonlinear Model Predictive Control* (Ch.5), Grüne and Pannek

Hanz Richter, Professor
Mechanical Engineering Department
Cleveland State University
Stability with Terminal Conditions

In this section we examine the use of terminal conditions (an equilibrium point or a larger invariant region) to guarantee NMPC stability.

Th. 4.11 in the previous handout uses some function $V(n, x)$ that must satisfy a relaxed DP inequality. Earlier (Th. 3.17), a DP equality had been established for the finite-horizon OCP, suggesting the use of $V_N(n, x)$ in Th. 4.11. However, the DP equality in Th. 3.17 has $V_{N-1}$ on the r.h.s., not $V_N$. When no terminal conditions are used, the cost function over $N$ steps is not smaller than the cost function over $N - 1$ steps, which is also valid for optimal costs:

$$V_N(n, x) \geq V_{N-1}(n, x)$$

The above inequality is the opposite of what would be needed to transform the DP equality into the desired inequality in Th. 4.11.

Terminal constraint sets are used to reverse the above inequality. Under a viability assumption for the terminal constraint set, recursive feasibility and stability will be established. Note that $\mathbb{X}$ does not need to be viable, only the terminal set $\mathbb{X}_0 \subseteq \mathbb{X}$ must be so.
Equilibrium Terminal Constraint

Consider the OCP with constant reference and terminal set $X_0 = x^*$:

$$\minimize_{u \in U^N_{X_0}(x_0)} J_N(x_0, u) = \sum_{k=0}^{N-1} l(x_u(k, x_0), u(k))$$

subject to

$$x_u(0, x_0) = x_0$$
$$x_u(k + 1, x_0) = f(x_u(k, x_0), u(k))$$

The terminal constraint implied by the notation $u \in U^N_{X_0}(x_0)$ requires

$$x_u(N, x_0) \in X_0$$

For an equilibrium constraint this is $x_u(N, x_0) = x^*$, where $f(x^*, u^*) = x^*$ for some admissible control $u^* \in U(x^*)$.

It is also assumed that $l(x^*, u^*) = 0$ (no cost associated to holding equilibrium).
Equilibrium Terminal Constraint

(Lemma 5.2) Under the above equilibrium assumptions, the following holds for $N \geq 2$:

i. Points which are viable in $N - 1$ steps (there is an admissible control sequence reaching $x^*$ in $N - 1$ steps) are also viable in $N$ steps. The obvious input sequence is obtained by appending $u^*$.

ii. Almost a re-statement of the above:

$$X_{N-1} \subseteq X_N$$

(Recursive feasibility of NMPC feedback, Lemma 5.3) Under the NMPC feedback law with terminal equilibrium constraint, the set $X_N$ is forward-invariant for the closed-loop system.

This result says that if $x \in X_N$, then $f(x, \mu_N(x))$ is again in $X_N$. This is the recursive feasibility property. We just need to check that the initial condition is viable, then we can be sure that the NMPC law will not halt because of infeasibility.
Equilibrium Terminal Constraint

(Lemma 5.4) Under the terminal equilibrium assumptions, for $N \geq 2$ and $x_0 \in X_{N-1}$, the optimal cost of the above OCPN satisfies

$$V_N(x_0) \leq V_{N-1}(x_0)$$

The proofs of Lemma 5.2 and Lemma 5.4 will be presented by students in class.

Note that reversal comes from the fact that it can become very expensive to exactly reach $x^*$ in a reduced number of steps, given the presence of constraints on $u$ and $x$. The inequality $V_N \geq V_{N-1}$ does not hold for the OCP with equilibrium terminal constraints.

Example: We compare the optimal costs associated with OCP of a double-integrator with and without equilibrium constraints.
Stability with Equilibrium Terminal Constraint

(Th. 5.5) Consider the NMPC feedback based on the above OCP. Assume that
\[ \exists \alpha_1, \alpha_2, \alpha_3 \in K_{\infty} \text{ such that} \]
\[ \alpha_1(|x|_{x^*}) \leq V_N(n, x) \leq \alpha_2(|x|_{x^*}) \]
\[ l(n, x, u) \leq \alpha_3(|x|_{x^*}) \]
for all \( x \in X, n \in \mathbb{N}_0, u \in U \).

Then the closed-loop system is asymptotically stable on \( \mathbb{X}_N \) and
\[ J_{\infty}^{cl}(x, \mu_N) \leq V_N(x) \]
for all \( x \in \mathbb{X}_N \).

Although each predicted trajectory must match \( x^* \) in \( N \) steps, this does not apply to the actual closed-loop trajectories.
Meeting the bounds on $V_N$ and $l$

With feasibility of $\mathbb{X}_0$, we just need to satisfy the above inequalities in $V_N$ and $l$ to obtain closed-loop asymptotic stability for any NMPC implementation.

(Proposition 5.7) i. Assume that $\exists \alpha_3 \in \mathcal{K}_\infty$ such that

$$l(x, u) \geq \alpha_3(|x|_x^*)$$

for all $x \in X, u \in U$. Then

$$V_N(x) \geq \alpha_3(|x|_x^*)$$

for all $x \in \mathbb{X}_N$.

Note that the commonly-used quadratic running costs

$$l(x, u) = \|x - x^*\|^2 + l_2(x, u)$$

with any $l_2(x, u) \geq 0$ satisfy the above.
Proposition 5.7...

ii. Assume \( f \) and \( l \) are continuous functions in \( X \times U \), where \( U \) is compact\(^1\) and \( \exists \beta_\nu(x^*) \subset X \) and a function \( \tilde{\alpha}_2 \in \mathcal{K}_\infty \) such that the following holds:
for each \( x \in \beta_\nu(x^*) \cap X \) there is an input \( u_x \in \mathbb{U}(x) \) with \( f(x, u_x) = x^* \) and
\[
l(x, u_x) \leq \tilde{\alpha}_2(|x|_{x^*})
\]
Then \( \exists \alpha_2 \in \mathcal{K}_\infty \) such that
\[
V_N(x) \leq \alpha_2(|x|_{x^*})
\]
for all \( x \in X_N \).

\(^1\)In \( \mathbb{R}^N \) “compact” is equivalent to closed and bounded
Example and ACADO coding

In class we examine Example 5.8 in detail, establish asymptotic stability. We also use this simple example to show how to code for NMPC using ACADO.
NMPC Stability with Terminal Region and Cost

A terminal equilibrium constraint may be too restrictive for some systems. For instance, there may be no admissible sequence bringing the state to equilibrium in a finite number of steps, that is we could encounter that $X_N = \emptyset$ for all $N \geq 1$.

Besides this fundamental issue, the requirement $x_u(N, x_0) = x^*$ is an exact equality constraint, and it may present numerical difficulties. Instead, it is possible to use a larger set $X_0$ which contains the equilibrium point $x^*$, in combination with a terminal weight.

Assumption (5.9):

i. The terminal constraint set is viable: for all $x \in X_0$ $\exists u_x \in U(x)$ such that $f(x, u_x) \in X_0$.

ii. The terminal cost function $F : X_0 \mapsto \mathbb{R}_0^+$ satisfies: for all $x \in X_0$ $u_x \in U(x)$ such that $f(x, u_x) \in U(x)$ and

$$F(f(x, u_x)) - F(x) \leq -l(x, u_x)$$
A set of technical lemmas paralleling the terminal equilibrium case holds. For details refer to G&P. We present stability theorem (5.13).

Consider the NMPC based on OCPNe with terminal constraint $x_{\mu_N}(N, x_0) \in X_0$, $x^* \in X_0$ and terminal cost $F(x_{\mu_N}(N, x_0))$, and assume conditions (5.9) (viability and Lyapunov-like condition on $F$) hold.$^2$

Assume also that $\exists \alpha_1, \alpha_2, \alpha_3 \in K_\infty$ satisfying the conditions of Th. 5.5. Then the closed-loop system is asymptotically stable on $X_N$ and $J_{cl}^\infty(x, \mu_N) \leq V_N(x) \forall x \in X_N$.

$^2$Note that the terminal equilibrium assumptions are a special case of (5.9), with $F = 0$ and $X_0 = x^*$
NMPC Stability with Terminal Region and Cost...

To apply this stability result we need to address the existence of $\alpha_1$, $\alpha_2$ and $\alpha_3$, and also find a method to construct $X_0$ and $F$. The first aspect is covered by Proposition 5.14:

i) If $\exists \alpha_3 \in \mathcal{K}_\infty$ such that $l(x,u) \geq \alpha_3(|x|_{x^*})$, $\forall x \in X$, $u \in U$, then

$$\exists V_N(x) \geq \alpha_3(|x|_{x^*}) \forall x \in X_N$$

ii) Suppose $f$ and $l$ are continuous on $X \times U$, $F$ is continuous on $X_0$ and $U$ is compact, and $\exists \tilde{\alpha}_2 \in \mathcal{K}_\infty$ such that

$$F(x) \leq \tilde{\alpha}_2(|x|_{x^*}) \forall x \in X_0 \cap \beta_n u(x^*)$$

then $\exists \alpha_2 \in \mathcal{K}_\infty$ such that $V_N(x) \leq \alpha_2(|x|_{x^*}) \forall x \in X_0$.

Note that under assumptions (5.9), it holds that $V_N(x) \leq V_{N-1}(x)$ for all $x \in X_{N-1}$ and $V_N(x) \leq F(x)$ for all $x \in X_0$. 
Terminal Set and Cost Construction - Linearization Approach

If the linearization of $f(x, u)$ at $(x^*, u^*)$ is stabilizable, an iterative, optimization-based approach can be followed to construct a suitable $X_0$ and $F$ combination (Chen and Allgöwer, '97).

We summarize the method in a separate handout and discuss code to solve the optimization problems.