The Laplace Transform

1. Laplace transforms are used to simplify handling of linear differential equations.

2. By taking the Laplace transform, a differential equation is converted to a simple algebraic equation in one variable (derivatives disappear and are replaced by powers of a new variable $s$: the Laplace variable)

3. Traditional differential equation solution techniques (variation of constants, particular vs. homogeneous solution etc.) are replaced by a single catch-all procedure (valid for linear ODEs with constant coefficients)

4. We will study the general method and use it for solving 1 DOF linear vibration problems.
Laplace Transform

- Defined by
  \[ \mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)\,dt \]

- \(s\) is the new variable.

- \(f(t)\) needs to be transformable: \(\int_0^\infty |f(t)|e^{-\sigma_1}\,dt < \infty\). Our \(f(t)\)'s will, no need to check.

- Inverse:
  \[ \mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds \]

- We don’t need to carry out the integrations. Just use a table of Laplace transform pairs.
Transform Properties

1. Linear operator:

\[ \mathcal{L}\{\alpha f_1 + \beta f_2\} = \alpha \mathcal{L}\{f_1\} + \beta \mathcal{L}\{f_2\} \]

2. Transform of a derivative:

\[ \mathcal{L}\left\{ \frac{d^k f(t)}{dt^k} \right\} = s^k F(s) - s^{k-1} f(0) - s^{k-2} f'(0) - \ldots - f^{(k-1)}(0) \]

In most cases the initial conditions are zero:
\[ f(0) = f'(0) = f''(0) = \ldots = 0, \text{ so} \]

\[ \mathcal{L}\left\{ \frac{d^k f(t)}{dt^k} \right\} = s^k F(s) \]

This property is the basis for operational calculus. Replaces derivatives by powers of \( s \), reducing differential equations to algebraic equations in the Laplace domain (\( s \) variable).
Example

Diff. eq:

\[ 2 \frac{d^3x}{dt^3} - \ddot{x} + 3\dot{x} + 6x = 4u \]

Laplace conversion, zero initial conditions:

\[ 2s^3X(s) - s^2X(s) + 3sX(s) + 6X(s) = 4U(s) \]

\[ (2s^3 - s^2 + 3s + 6)X(s) = 4U(s) \]

NOTES:

1. \( X(s) \) and \( U(s) \) are the transforms of \( x(t) \) and \( u(t) \). They are different animals, don’t mix them.

2. We use block letters for Laplace transforms, small letters for time functions.

3. Sometimes we drop the \((s)\) argument: \((2s^3 - s^2 + 3s + 6)X = 4U\).
Example: Nonzero Initial Conditions

Diff. eq:

\[ 2 \frac{d^3 x}{dt^3} - \ddot{x} + 3\dot{x} + 6x = 4u \]

Laplace conversion, nonzero initial conditions:

Powers of \( s \) decrease

\[ 2 \left[ s^3 X(s) - s^2 x(0) - s\dot{x}(0) - \ddot{x}(0) \right] \]

Derivatives in \( x(0) \) increase

\[ - \left[ s^2 X(s) - sx(0) - \dot{x}(0) \right] \]

\[ + 3 \left[ sX(s) - x(0) \right] \]

\[ + 6X(s) = 4U(s) \]
Solving Linear Constant-Coefficient ODEs with Laplace

1. The diff.eq., initial conditions and forcing function $u(t)$, $F(t)$, etc. are given.
2. Take the Laplace transform of the diff.eq. taking initial conditions into account.
3. Use a Laplace table to find $U(s)$ from $u(t)$
4. Solve for $X(s)$ (or whatever the unknown function is). The result is some expression in $s$.
5. If possible, locate an entry in the Laplace table matching the above solution and read the corresponding $x(t)$.
6. If necessary, a partial fraction expansion must be computed prior to using the table.
Example: Direct Lookup

Differential equation:

\[ \ddot{x} + x = u \]

Assume that \( u(t) = 2t \) and use zero initial conditions.

1. Transform the equation: \( (s^2 + 1)X(s) = U(s) \)
2. Look up the transform of \( u(t) \): \( U(s) = \frac{2}{s^2} \)
3. Substitute and solve: \( X(s) = \frac{2}{s^2(s^2 + 1)} \)
4. Look up in reverse: \( x(t) = 2(t - \sin(t)) \).
PFE Decomposition: Real Poles

- Suppose $X(s) = \frac{n(s)}{d(s)}$, with $X(s)$ *strictly proper* (degree of denominator greater than degree of numerator).

- The roots of $d(s) = 0$ are the *poles*.

- To every simple real pole at $s = p$ there corresponds a single term of the form

$$\frac{a}{s - p}$$

where

$$a = \lim_{s \to p} (s - p) G(p)$$

- To every multiple real pole of multiplicity $k \geq 2$ there correspond $k$ terms:

$$\frac{c_1}{s - p} + \frac{c_2}{(s - p)^2} + \ldots + \frac{c_k}{(s - p)^k}$$
To every simple pair of complex conjugate poles at \( s = \alpha \pm \beta i \) there correspond two terms of the form

\[
\frac{A(s - \alpha)}{(s - \alpha)^2 + \beta^2} + \frac{B}{(s - \alpha)^2 + \beta^2}
\]

The quantities \( a, c_i, A \) and \( B \) are called residues.

Matlab can calculate the residues, but can give complex values.

We’ll skip the case of multiple complex conjugate poles. Rarely found and may be solved using Matlab.
Obtaining the PFE - Recipe

- Find the poles and characterize them as real or complex, single or multiple.
- Determine the structure of the decomposition.
- If single real poles are found, find the residues now.
- Multiply through and equate the coefficients of the numerator to those of the original $X(s)$
- Solve a linear system of equations.
- **NOTE**: If you found some residues for single real poles, you will have more equations than unknowns. Use the extra equations for verification.
Example: PFE Required

Suppose \( u(t) \) is a unit step input in the previous example, and the initial conditions are

\[
\ddot{x}(0) = 1, \quad \dot{x}(0) = -1, \quad x(0) = 2
\]

From the table: \( U(s) = 1/s \). Then

\[
(2s^3 - s^2 + 3s + 6)X(s) = \frac{4}{s} + 2x(0)s^2 + (2\dot{x}(0) - x(0))s + 2\ddot{x}(0) - \dot{x}(0) + 3x(0)
\]

Substituting the initial conditions:

\[
(2s^3 - s^2 + 3s + 6)X(s) = \frac{4}{s} + 4s^2 - 4s + 9
\]

\[
X(s) = \frac{4s^3 - 4s^2 + 9s + 4}{s(2s^3 - s^2 + 3s + 6)}
\]
Example...

Find the characteristic roots by making the denominator $= 0$.

$$s = 0, \quad s = -1, \quad s = 0.75 \pm 1.5612i$$

Factor denominator polynomial into quadratic and linear polynomials:

$$2(s^3 - 0.5s^2 + 1.5s + 3) = 2(s + 1)(s^2 - 1.5s + 3)$$

**Note:** With Matlab, knowing that $(s + 1)$ is a factor, divide using `deconv([1 -0.5 1.5 3],[1 1])`.

The form of the decomposition will be

$$\frac{a_0}{s} + \frac{a_1}{s + 1} + \frac{A(s - \alpha)}{(s - \alpha)^2 + \beta^2} + \frac{B}{(s - \alpha)^2 + \beta^2}$$

where $\alpha = 0.75$ and $\beta = 1.5612$. **Note:** Denominator is just the quadratic factor $s^2 - 1.5s + 3$. 
Example...

Find the residues for the real roots: $a_0 = 2/3$ and $a_1 = 13/11$
Recombine the PFE by hand or by Matlab (Symbolic Toolbox needed):

```matlab
>> syms s A B
>> expr=2/3/s+13/11/(s+1)+A*(s-0.75)/(s^2-1.5*s+3)+B/(s^2-1.5*s+3)
>> collect(expr,s)
```
Carefully match coefficients

\[
A + \frac{244}{132} = 2
\]
\[
33A/132 + B - \frac{278}{132} = -2
\]
\[
B - \frac{99A}{132} + \frac{600}{132} = \frac{9}{2}
\]
\[
\frac{264}{132} = 2
\]
Example...

Check that redundant equations are consistent and solve: $A = 20/132$, $B = 9/132$
Recombine the PFE with numerical values and check that it equals the original expression:

```
>> A = 20/132; B = 9/132;
>> factor(eval(expr))
```

Look up the partial terms in the Laplace table

\[
\begin{align*}
\frac{2}{3s} & \rightarrow 2/3 \\
\frac{13}{11(s + 1)} & \rightarrow \frac{13}{11}e^{-t} \\
\frac{(20/132)(s - 0.75)}{(s - 0.75)^2 + 1.5612^2} & \rightarrow (20/132)e^{0.75t} \cos(1.5612t) \\
\frac{(9/132)}{(s - 0.75)^2 + 1.5612^2} & \rightarrow (3/132)e^{0.75t} \sin(1.5612t)
\end{align*}
\]
Example...

Overall solution:

\[ x(t) = \frac{2}{3} + \frac{13}{11} e^{-t} + \frac{20}{132} e^{0.75t} \cos(1.5612t) + \frac{3}{132} e^{0.75t} \sin(1.5612t) \]

Check by plotting \( x(t) \) for \( t \geq 0 \) and comparing with an ode45 solution. **Do this as a way to practice plotting and diff.eq. solutions using Matlab**

**Notes**

1. Problems requiring PFE will be assigned as homework or take-home exam portions only.

2. The in-class portion of the midterm exam will include a non-PFE diff. eq. solution.

3. The general method allows you to find the solution to *any* linear 1-DOF vibration problem with constant mass, stiffness and damping.