MCE 647 - Robotics Homework 3

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An RPR robot is considered, that can be described by the following summary table:

Link	θ	d	a	α
1	q_1	0	0	0
2	0	q_2	0	$-\frac{\pi}{2}$
3	q_3	d	0	0

Table 1: D-H Summary Table

A point P is located at $P^3 = [L, 0, 0]^T$, and it is assumed that both L and d are positive. The goal is to analyze several aspects of the velocity kinematics of the robot, including finding the velocity Jacobians, finding specific velocities with respect to certain joint variable values, finding singular configurations, analyzing the manipulability ellipsoid, and finding the Yoshikawa measure.

1 Problem 1

The linear and and angular velocity jacobians, $J_v(q)$ and $J_\omega(q)$, respectively, are shown below:

$$J_v(q) = \begin{bmatrix} -d\cos(q_1) - L\sin(q_1)\cos(q_3) & 0 & -L\cos(q_1)\sin(q_3) \\ -d\sin(q_1) + L\cos(q_1)\cos(q_3) & 0 & -L\sin(q_1)\sin(q_3) \\ 0 & 1 & -L\cos(q_3) \end{bmatrix} \tag{1}$$

$$J_\omega(q) = \begin{bmatrix} 0 & 0 & -\sin(q_1) \\ 0 & 0 & \cos(q_1) \\ 1 & 0 & 0 \end{bmatrix} \tag{2}$$

2 Problem 2

The world velocities can now be found for any feasible set of joint variable values and velocities. When $q = [0, 1, -\frac{\pi}{2}]^T$ and $\dot{q} = [0, 0, -1]^T$, then P will follow the world velocity vector $v = [-L, 0, 0]^T$ and will be rotating as $\omega = [0, -1, 0]^T$. This can be verified by sketching this configuration, as shown in fig. 1:

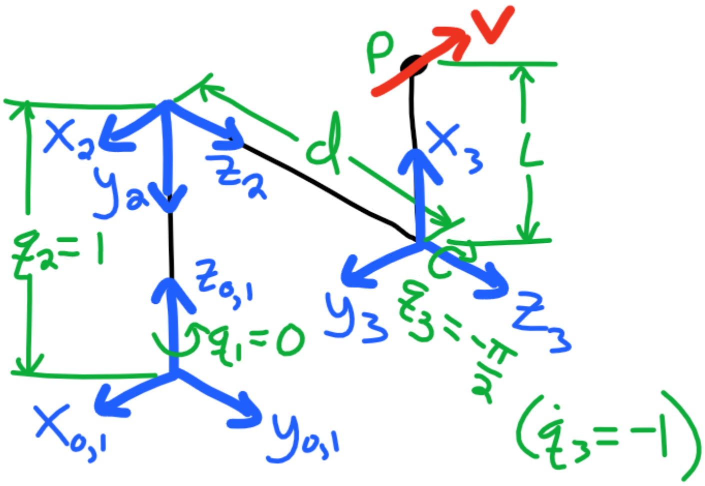

Figure 1: Configuration 1

When $q = [\frac{\pi}{2}, 1, \frac{\pi}{2}]^T$ and $\dot{q} = [0, 0, 1]^T$, then P will follow the world velocity vector $v = [0, -L, 0]^T$ and will be rotating as $\omega = [-1, 0, 0]^T$. This can be verified by sketching this configuration, as shown in fig. 2:

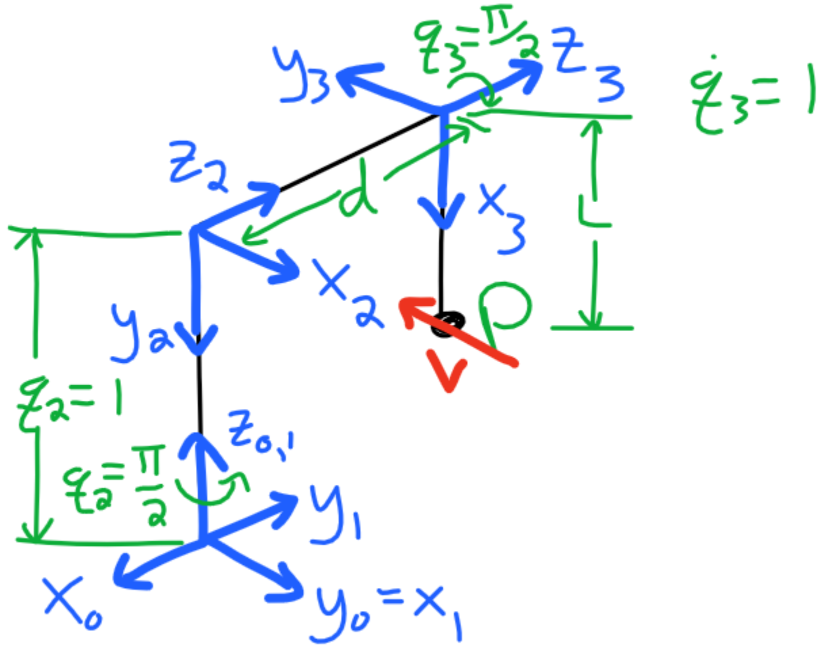

Figure 2: Configuration 2

3 Problem 3

Singular velocity configurations occur when the velocity jacobians are reduced in rank from their maximum possible rank.

Let us first analyze the linear velocity jacobian. Because J_v found in eq. 1 is of full rank, we can look for cases when its determinant is zero. Because the determinant of J_v is $-L^2 \sin(2q_3)/2$, singular points will occur when $q_3 = n\frac{\pi}{2}$ for any integer n. This will produce a few different cases, namely $q_3 = 2\pi n$,

$q_3 = 2\pi n + \pi$, $q_3 = 2\pi n + \frac{\pi}{2}$, and $q_3 = 2\pi n + \frac{3\pi}{2}$. We will analyze these cases two at a time, starting with the first two:

$$J_v(q_3 = 2\pi n) = \begin{bmatrix} -d\cos(q_1) - L\sin(q_1) & 0 & 0 \\ -d\sin(q_1) + L\cos(q_1) & 0 & 0 \\ 0 & 1 & -L \end{bmatrix} \tag{3}$$

$$J_v(q_3 = 2\pi n + \pi) = \begin{bmatrix} -d\cos(q_1) + L\sin(q_1) & 0 & 0 \\ -d\sin(q_1) - L\cos(q_1) & 0 & 0 \\ 0 & 1 & L \end{bmatrix} \tag{4}$$

In these two configurations, the x and y components of the velocity lose dependence on $\dot{q_3}$. This is due to the fact that, relative to $\dot{q_3}$, P can only move along the z_0 direction. This becomes obvious when one notices that, when $q_3 = 2\pi n$ or $q_3 = 2\pi n + \pi$, the x_3 axis is parallel to the (x_0, y_0) plane, and P WRT q_3 can only rotate about the z_3 axis.

As long as L and d are greater than zero, it is not possible to reduce the rank of J_v below 2 for either of these configurations. In order to do so, the entire first column would need to be comprised of zeros, which is only satisfied by $\tan(q_1) = -\frac{d}{L} = \frac{L}{d}$ for the first case and $\tan(q_1) = \frac{d}{L} = -\frac{L}{d}$ for the second case. It is impossible to satisfy either of these sets of equations.

The second two singular configurations are brought about for slightly different reasons:

$$J_v(q_3 = 2\pi n + \frac{\pi}{2}) = \begin{bmatrix} -d\cos(q_1) & 0 & -L\cos(q_1) \\ -d\sin(q_1) & 0 & -L\sin(q_1) \\ 0 & 1 & 0 \end{bmatrix} \tag{5}$$

$$J_v(q_3 = 2\pi n + \frac{3\pi}{2}) = \begin{bmatrix} -d\cos(q_1) & 0 & L\cos(q_1) \\ -d\sin(q_1) & 0 & L\sin(q_1) \\ 0 & 1 & 0 \end{bmatrix} \tag{6}$$

In these two configurations, the x and y velocities of P are linearly dependent on each other. This is due to the fact that, when $q_3 = 2\pi n + \frac{\pi}{2}$ or $q_3 = 2\pi n + \frac{3\pi}{2}$, the x_3 axis is normal to the (x_0, y_0) plane. At this point, P has velocity components purely in the (x_0, y_0) plane, and the (x_0, y_0) components of the velocity are related (by trigonometry) solely to the angle q_1, and are therefore directly related to each other.

As long as L and d are greater than zero, it is not possible to reduce the rank of J_v below 2 for either of these configurations. In order to do so, both the first and third columns would need to be zero, which is impossible, as it is not possible to satisfy $\cos(q_1) = \sin(q_1) = 0$.

The angular velocity jacobian is already rank deficient ($\text{rank}(J_\omega) = 2$). It is not possible to reduce the rank of J_ω further, as it is not possible to satisfy $\cos(q_1) = \sin(q_1) = 0$, as described previously.

4 Problem 4

It is now desired to find the manipulability ellipsoid for particular configurations. Let us first consider the configuration where $q = [\frac{\pi}{4}, 1, \frac{\pi}{2}]^T$. Assuming $d = L = 1$, we can find the magnitudes and directions of the principal axes of the ellipsoid. Eq. 7 describes the normalized (x, y, z) directions of the major axes (in order of decreasing magnitude) and eq. 8 lists the corresponding axis magnitudes:

$$\begin{bmatrix} \frac{U_1}{\|U_1\|} & \frac{U_2}{\|U_2\|} & \frac{U_3}{\|U_3\|} \end{bmatrix} = \begin{bmatrix} -0.7071 & 0 & 0 \\ -0.7071 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} \|U_1\| & \|U_2\| & \|U_3\| \end{bmatrix} = \begin{bmatrix} 1.4142 & 1 & 0 \end{bmatrix} \tag{8}$$

This configuration and its manipulability ellipsoid are shown in fig. 3 (note that one of the axes has zero magnitude, which is reflected in the figure). The red axis in the ellipse is the axis with the largest magnitude, the blue axes (one of which is a point in this configuration) denote the smaller axes. The black bar represents the robotic manipulator, ending at point P. Green indicates points on the manipulability ellipse.

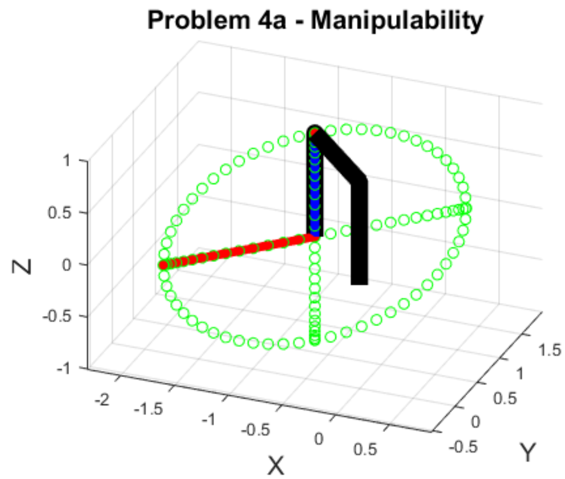

Figure 3: Configuration A

Now let us consider the configuration where $q = [\frac{\pi}{2}, 1, \frac{-\pi}{4}]^T$. Assuming $d = L = 1$, we can find the magnitudes and directions of the principal axes of the ellipsoid. Eq. 9 describes the normalized (x, y, z) directions of the major axes (in order of decreasing magnitude) and eq. 10 lists the corresponding axis magnitudes:

$$\begin{bmatrix} \frac{U_1}{\|U_1\|} & \frac{U_2}{\|U_2\|} & \frac{U_3}{\|U_3\|} \end{bmatrix} = \begin{bmatrix} -0.3636 & -0.4293 & -0.2499 \\ -0.8169 & -0.2059 & 0.3279 \\ 0.4477 & -0.8794 & 0.9110 \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} \|U_1\| & \|U_2\| & \|U_3\| \end{bmatrix} = \begin{bmatrix} 1.4607 & 1.0430 & 0.5278 \end{bmatrix} \tag{10}$$

This configuration and its manipulability ellipsoid are shown in fig. 4. The color scheme is identical to the color scheme of fig. 3.

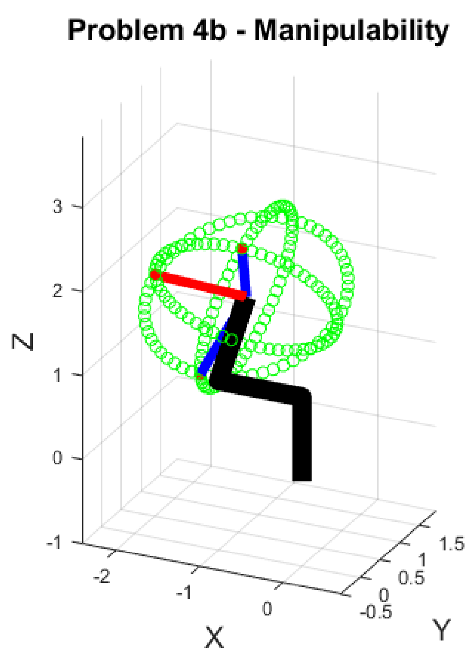

Figure 4: Configuration B

5 Problem 5

The Yoshikawa manipulability measure can be solved for symbolically in Matlab. For this manipulator, the Yoshikawa manibulability (calculated using only the linear velocity jacobian) can be described by eq. 11:

$$\mu(q) = \frac{L^2\sqrt{2}}{4}\sqrt{1-\cos(4q_3)} \tag{11}$$

It can then be shown that the maximum manipulability occurs when $\cos(4q_3) = -1$, or at $q_3 = \frac{\pi}{4} \pm \frac{\pi}{2}n$.

6 Problem 6

If J_v and J_ω are combined (stacked together) to create the full velocity jacobian J and used to calculate the Yoshikawa manipulability, then the manipulability can be described by eq. 12:

$$\mu(q) = \sqrt{L^2 - \frac{L^4}{8}(\cos(4q_3)-1) + d^2 + 1} \tag{12}$$

It should be noted that this measure depends only on q_3. This occurs because q_1 and q_2 merely shift the position of the ellipsoid without altering the manipulability.

7 Problem 8

Based upon the results from problem 6, if L and d are taken to be 1, then the Yoshikawa manipulability is reduced to the following function described by eq. 13:

$$\mu(q) = \sqrt{\frac{1}{8}(25 - \cos(4q_3))} \tag{13}$$

It can then be shown that the maximum manipulability occurs when $\cos(4q_3) = -1$, or at $q_3 = \frac{\pi}{4} \pm \frac{\pi}{2}n$. Note that these values of q_3 also produce a maximum for all L and d (see eq. 12).