1. DH table:

<table>
<thead>
<tr>
<th>Link</th>
<th>θ</th>
<th>d</th>
<th>a</th>
<th>α</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>θ₁</td>
<td>2</td>
<td>0</td>
<td>-1/2</td>
</tr>
<tr>
<td>2</td>
<td>θ₂</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

1) \( H_1^0 = \overbrace{\text{Rot}_{z,-\frac{\pi}{2}}}^{\text{Rot}_{z,\frac{\pi}{2}}} \overbrace{\text{Trans}_{x,2}}^{\text{Trans}_{x,1}} \overbrace{\text{Rot}_{x,\theta_1}}^{\text{Rot}_{x,\theta_1}} \)

\[ \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

Multiply: \( H_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \)

\( c_1 \equiv \cos(\theta_1) \quad s_1 \equiv \sin(\theta_1) \)

\( H_2^1 = \overbrace{\text{Rot}_{z,\theta_2}}^{\text{Rot}_{z,\theta_2}} \overbrace{\text{Trans}_{z,1}}^{\text{Trans}_{z,1}} \overbrace{\text{Trans}_{x,\theta_1}}^{\text{Trans}_{x,\theta_1}} = \begin{bmatrix} c_2 & -s_2 & 0 & c_2 \\ s_2 & c_2 & 0 & s_2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \)

\( H_2^0 = H_1^0 \cdot H_2^1 = \begin{bmatrix} c_1 c_2 & -c_1 s_2 & -s_1 & c_1 c_2 - s_1 \\ c_2 s_1 & -s_2 & -c_1 & c_1 c_2 s_1 \\ -s_2 & -c_2 & 0 & 1 - s_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \)
For \( q_1 = \pi/4 \) and \( q_2 = 0 \) we have

\[
H_2^o = \begin{bmatrix}
\sqrt{2}/2 & 0 & -\sqrt{2}/2 \\
\sqrt{2}/2 & 0 & \sqrt{2}/2 \\
0 & -1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

\[
O_2^o
\]

\[
Z_2^o
\]

\[
\theta^e
\]

\[
\begin{align*}
J_{v_1} &= Z_0 \times \hat{z}_2 = \\
&= \begin{bmatrix}
-c_2 s_1 & -c_2 s_1 & 0 \\
c_2 c_1 - s_1 & c_2 c_1 - s_1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{align*}
\]

\[
J_{v_2} = \hat{z}_1 \times (O_2 - O_1) = \begin{bmatrix}
-c_1 s_2 & -s_1 s_2 & -c_2 \\
-s_1 s_2 & -s_1 s_2 & -c_2 \\
-c_1 s_2 & -s_1 s_2 & -c_2
\end{bmatrix}
\]

\[
J_w = \begin{bmatrix}
z_0 & \hat{z}_1
\end{bmatrix} = \begin{bmatrix}
0 & -s_1 \\
0 & c_1 \\
1 & 0
\end{bmatrix}
\]
\[ \mathbf{J}_V = \begin{bmatrix} -c_1 - c_2 s_1 & -c_1 s_2 \\ c_1 c_2 - s_1 & -s_1 s_2 \\ 0 & -c_2 \end{bmatrix} \]

(647) We can find a singularity when \( c_2 = 0 \):

\[ \mathbf{J}_V = \begin{bmatrix} -c_1 & -c_1 s_2 \\ -s_1 & -s_1 s_2 \\ 0 & 0 \end{bmatrix} \quad \text{col} \, 2 = s_2 \times \text{col} \, 1 \quad \Rightarrow \quad \text{rank} \, (\mathbf{J}_V) = 1 \]

Take \( q_2 = n\pi/2 \), \( n = \pm 1, \pm 3 \ldots \)

(747) For the columns of \( \mathbf{J}_V \) to be dependent,

it is necessary that \( c_2 = 0 \). From above,

this implies \( \text{rank} \, (\mathbf{J}_V) = 1 \).

Then \( \text{rank} \, (\mathbf{J}_V) < 2 \iff c_2 = 0 \quad (q_2 = n\pi/2) \)

\[ \times \quad \times \]

② From notes, the Jacobian is

\[ \mathbf{J}_V = \begin{bmatrix} -s_{12} - s_1 & -s_{12} \\ c_{12} + c_1 & c_{12} \\ - & - & - & 1 \end{bmatrix} \]

\( s_{12} \equiv \sin(q_1 + q_2) \)

\( c_{12} \equiv \cos(q_1 + q_2) \)

\[ \det(\mathbf{J}_V) = -s_1 c_{12} + c_1 s_{12} = s_2 \quad \star \text{NOTE} \star \]
The Jacobian can be decomposed as:

\[ J_r = U \Sigma V^T. \]

From the figure, we know \( \sigma_1 = \sqrt{5} \) and the corresponding column of \( U \) is the unit vector in the given direction:

\[ U(:, 1) = \begin{bmatrix} \cos 30^\circ \\ \sin 30^\circ \end{bmatrix} \]

\[ U(:, 1) = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix} \]

Since \( U \) must be orthogonal:

\[ U(:, 2) = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \]

We also know that \( \sigma_2 = 0 \), corresponding to the null vector \( V(:, 2) \).

Finally, the 2-link planar manipulator has singularities only when \( \theta_2 = 0, \pm \pi, \pm 2\pi \ldots \) (from previous note). 

Let \( V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & 2/\sqrt{5} \end{bmatrix} \)

Then:

\[ U \Sigma = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{15}}{2} & 0 \\ \frac{\sqrt{5}/2}{2} & 0 \end{bmatrix} \]

and \( U \Sigma V^T = \begin{bmatrix} \frac{-\sqrt{15} v_{11}}{2} & -\frac{\sqrt{15} v_{21}}{2} \\ \frac{\sqrt{5}}{2} v_{11} & \frac{\sqrt{5}}{2} v_{21} \end{bmatrix} \) \( \cdots \cdots \) (2)
Equating (1) and (2):

\[
\begin{bmatrix}
\frac{-\sqrt{15}}{2} v_{11} & \frac{-\sqrt{15}}{2} v_{21} \\
\frac{\sqrt{5}}{2} v_{11} & \frac{\sqrt{5}}{2} v_{21}
\end{bmatrix}
= \begin{bmatrix}
-2s_1 & -s_1 \\
2c_1 & c_1
\end{bmatrix}
\]  \(\tag{3}\)

and from:

\[
\begin{bmatrix}
-2s_1 & -s_1 \\
2c_1 & c_1
\end{bmatrix}
\begin{bmatrix}
v_{12} \\
\frac{2}{\sqrt{5}}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

That is,

\[
\begin{cases}
-s_1 \left(2v_{12} + \frac{2}{\sqrt{5}}\right) = 0 \\
c_1 \left(2v_{12} + \frac{2}{\sqrt{5}}\right) = 0
\end{cases}
\]

but \(s_1 \neq 0\), because if \(s_1 = 0\) we would have \(v_{11} = 0\) and \(v_{21} = 0\) from (1), and \(V\) wouldn't be orthogonal.

Then:

\[
v_{12} = -\frac{1}{\sqrt{5}}
\]

(we can reach the same conclusion by seeing that \(s_1\) can't be zero)

\[
\Rightarrow V(\psi, 2) = \begin{bmatrix}
-\frac{1}{\sqrt{5}} \\
\frac{2}{\sqrt{5}}
\end{bmatrix}
\]

From (3) we see that the second column of \(UZV^T\) is \(\frac{1}{2}\) the first.
\[ \begin{bmatrix} -\frac{\sqrt{15}}{2} v_2 \\ \frac{\sqrt{5}}{2} v_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\sqrt{5} \frac{v_1}{2} \\ \frac{v_5}{2} \frac{v_4}{4} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{v_4}{4} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} v_1 \end{bmatrix} \\

By orthogonality of \( V \): \( v_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \)

so \( V = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \)

so \( J_r = U \Sigma V^T = \begin{bmatrix} -\sqrt{3} & -\sqrt{3}/2 \\ 1 & 1/2 \end{bmatrix} = \begin{bmatrix} -2\sigma_1 & -\sigma_1 \\ 2\sigma_1 & \sigma_1 \end{bmatrix} \)

\[ \Rightarrow \begin{cases} \sigma_1 = \sqrt{3}/2 \\ \sigma_1 = 1/2 \end{cases} \quad \Rightarrow \quad \theta_1 = 30^\circ \]

(3)

a.) This can happen if \( J_\omega(q) \) is singular at some \( q \). Then \( \omega = J_\omega(q) \bar{q} \)

so taking \( \bar{q} \in \text{null}(J_\omega) \) will give \( \omega = 0 \) (instantaneous translation)

b. A Cartesian 3-link robot has \( J_r = I \) \( \Rightarrow \theta_1 = 1 \) (isotropic)
There may be an infinity of solutions, depending on the location of the target point and the link lengths.