HW1 – Sp 12 Solution by Baykut, Richter and Jagodnik

MCE 747

ROBOT DYNAMICS & CONTROL

HOMEWORK I PART 2

MERT BAYKUT

Problem 2.14:

$$R_{y,\frac{\pi}{2}} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \text{ and } R_{x,\frac{\pi}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

The first rotation is done by rotating the frame with respect to the world frame and the second rotation is done with respect to the fixed frame, therefore, we have to multiply the rotation matrices in reverse order.

$$R = R_{x,\frac{\pi}{2}} R_{y,\frac{\pi}{2}} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

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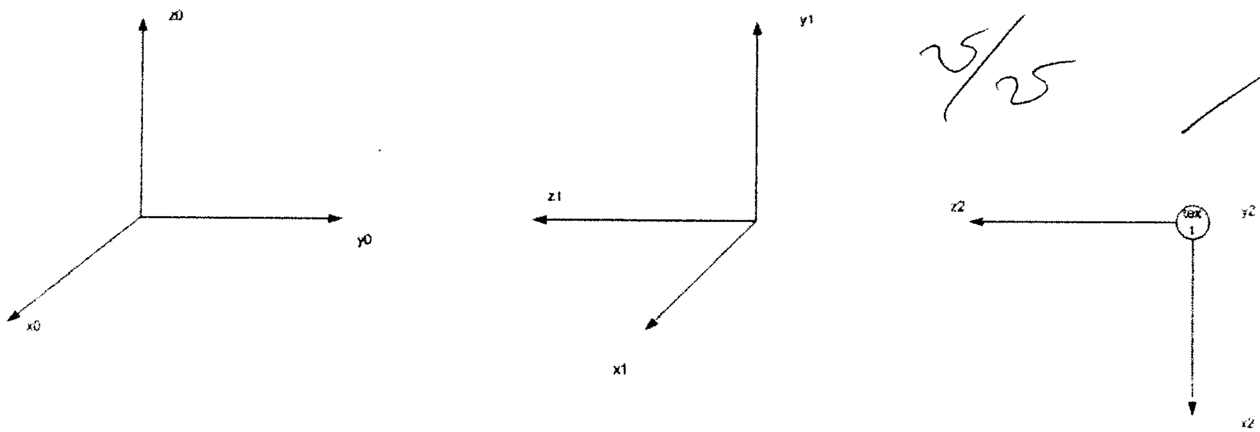

Please notice for the last frame y_2 is pointing outside of the paper.

2-19 — Solution by H. Richter

A rotation matrix is defined by 2 properties:

$\|Ax\| = \|x\|$ (transformed vectors do not change length)

$\det A = 1$

We need to show that $\lambda = 1$ is an eigenvalue of A.

Let v be an eigenvector of A, then

$$\|Av\| = \|\lambda v\| = |\lambda| \, \|v\|$$

But $\|Av\| = \|v\|$, so $|\lambda| \, \|v\| = \|v\|$.

Since $\|v\| \neq 0$, this implies $|\lambda| = 1$. for all eigenvalues.

f A is 3×3, $\det A = \lambda_1 \lambda_2 \lambda_3$, where λ_i are the eigenvalues of A. So $\lambda_1 \lambda_2 \lambda_3 = 1$.

Since λ's are the solutions of a cubic polynomial, 2 cases can occur:

i) λ_1, λ_2 and λ_3 are real

ii) One λ is real and the other 2 are complex conjugates.

n case i), at least one λ must be positive to have $\lambda_1 \lambda_2 \lambda_3 = 1$, so one λ must be 1

In case ii), suppose λ_1 is real and $\lambda_2 = \overline{\lambda_3}$ (complex conjugates). Then $\lambda_1 \lambda_2 \lambda_3 = \lambda_1 \lambda_2 \overline{\lambda_2} = \lambda_1 |\lambda_2$

So $\lambda_1 |\lambda_2|^2 = 1$, but $|\lambda_2|^2 = 1$, so $\lambda_1 = 1$.

//

— x —

(A similar proof can be followed for 2×2)

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MCE/EEC 647/747: Robot Dynamics and Control
Homework 1 (Part 2) – Spring 2012

PART 2 DUE: 02-08-12
Solve the following problems from SHV: 2-14, 2-19, and **2-22**, and 2-40.

Homework 1, Part 2

Spong, Mark W., Seth Hutchinson, M. Vidyasagar. Robot Modeling and Control. Wiley, 2006.

Solve Problem 2-22

Compute the rotation matrix given by the product

$$R_{x,\theta}R_{y,\varphi}R_{z,\pi}R_{y,-\varphi}R_{x,-\theta}$$

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Solution:

– We are to compute the composite rotation matrix given by the product of five (5) three-dimensional (3D) rotation matrices $R_{x,\theta}R_{y,\varphi}R_{z,\pi}R_{y,-\varphi}R_{x,-\theta}$ where the individual rotations are defined by

$$R_{x,\theta}=\begin{bmatrix}1&0&0\\0&c\theta&-s\theta\\0&s\theta&c\theta\end{bmatrix},\ R_{y,\varphi}=\begin{bmatrix}c\varphi&0&s\varphi\\0&1&0\\-s\varphi&0&c\varphi\end{bmatrix},\ R_{z,\pi}=\begin{bmatrix}\cos\pi&-\sin\pi&0\\\sin\pi&\cos\pi&0\\0&0&1\end{bmatrix},\ R_{y,-\varphi}=\begin{bmatrix}c(-\varphi)&0&s(-\varphi)\\0&1&0\\-s(-\varphi)&0&c(-\varphi)\end{bmatrix},\ R_{x,-\theta}=\begin{bmatrix}1&0&0\\0&c(-\theta)&-s(-\theta)\\0&s(-\theta)&c(-\theta)\end{bmatrix}$$

– Due to the complexity and unwieldy calculation of multiplying five (5), 3×3 matrices, MATLAB shall be employed... (see pages 2 and 3 for MATLAB commands).

The rotation matrix given by the product above is

$$=\begin{bmatrix}
\cos\pi(\cos^2(\varphi))+\sin^2(\varphi) & -\sin(\theta)(\cos(\varphi)*\sin(\varphi)-\cos(\pi)*\cos(\varphi)\sin(\varphi))-\cos(\varphi)*\cos(\theta)*\sin(\pi) & \cos\theta(\cos(\varphi)\sin(\varphi)-\cos(\pi)\cos(\varphi)*\sin(\varphi))-\cos(\varphi)*\sin(\pi)*\sin(\theta) \\
\cos(\varphi)[\cos(\theta)\sin(\pi)+\cos(\pi)\sin(\varphi)\sin(\theta)]-\cos(\varphi)\sin(\varphi)\sin(\theta) & \cos(\theta)[\cos(\pi)\cos(\theta)-\sin(\pi)\sin(\varphi)\sin(\theta)]+\sin(\theta)[\sin(\theta)\cos^2(\varphi)+\sin(\varphi)*(\cos(\theta)\sin(\pi)+\cos(\pi)*\sin(\varphi)+\sin(\theta)]]] & \sin(\theta)[\cos(\pi)\cos(\theta)-\sin(\pi)\sin(\varphi)\sin(\theta)]-\cos(\theta)*[\sin(\theta)\cos^2(\varphi)+\sin(\varphi)[\cos(\theta)\sin(\pi)+\cos(\pi)\sin(\varphi)\sin(\theta)]]] \\
\cos(\varphi)[\sin(\pi)\sin(\theta)-\cos(\pi)\cos(\theta)\sin(\varphi)]+\cos(\varphi)\cos(\theta)\sin(\varphi) & \cos(\theta)[\cos(\pi)\sin(\theta)+\cos(\theta)*\sin(\pi)\sin(\varphi)]-\sin(\theta)[\cos^2(\varphi)\cos(\theta)-\sin(\varphi)[\sin(\pi)\sin(\theta)-\cos(\pi)\cos(\theta)\sin(\varphi)]] & \cos(\theta)[\cos^2(\varphi)\cos(\theta)\sin(\varphi)[\sin(\pi)\sin(\theta)-\cos(\pi)\cos(\theta)\sin(\varphi)]]+\sin(\theta)[\cos(\pi)\sin(\theta)+\cos(\theta)\sin(\pi)\sin(\varphi)]]
\end{bmatrix}$$

3×3

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Homework 1, Part 2

Problem #2-22 continued...

```
EDU>>
EDU>>
EDU>>
EDU>>
EDU>>
EDU>> Rx=[1 0 0;0 cos(th) -sin(th);0 sin(th) cos(th)]

Rx =

[ 1,        0,        0]
[ 0, cos(th), -sin(th)]
[ 0, sin(th),  cos(th)]

EDU>> Ry=[cos(phi) 0 sin(phi);0 1 0;-sin(phi) 0 cos(phi)]

Ry =

[  cos(phi), 0, sin(phi)]
[         0, 1,        0]
[ -sin(phi), 0, cos(phi)]

EDU>> Rz=[cos(ga) -sin(ga) 0;sin(ga) cos(ga) 0;0 0 1]

Rz =

[ cos(ga), -sin(ga), 0]
[ sin(ga),  cos(ga), 0]
[       0,        0, 1]

EDU>> Ryn=[cos(-phi) 0 sin(-phi);0 1 0;-sin(-phi) 0 cos(-phi)]

Ryn =

[ cos(phi), 0, -sin(phi)]
[        0, 1,         0]
[ sin(phi), 0,  cos(phi)]

EDU>> Rxn=[1 0 0;0 cos(-th) -sin(-th);0 sin(-th) cos(-th)]
```

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Homework 1, Part 2

Problem #2-22 continued...

```
Rxn =

[ 1,        0,        0]
[ 0,  cos(th), sin(th)]
[ 0, -sin(th), cos(th)]

EDU>> (Rx)*(Ry)*(Rz)*(Ryn)*(Rxn)

ans =

[                                                cos(ga)*cos(phi)^2 + sin(phi)^2,↙
- sin(th)*(cos(phi)*sin(phi) - cos(ga)*cos(phi)*sin(phi)) - cos(phi)*cos(th)*sin(ga),↙
cos(th)*(cos(phi)*sin(phi) - cos(ga)*cos(phi)*sin(phi)) - cos(phi)*sin(ga)*sin(th)]
[ cos(phi)*(cos(th)*sin(ga) + cos(ga)*sin(phi)*sin(th)) - cos(phi)*sin(phi)*sin(th), cos(th)*(cos(ga)*cos(th) - sin(ga)*sin(phi)↙
*sin(th)) + sin(th)*(sin(th)*cos(phi)^2 + sin(phi)*(cos(th)*sin(ga) + cos(ga)*sin(phi)*sin(th))), sin(th)*(cos(ga)*cos(th) - sin↙
(ga)*sin(phi)*sin(th)) - cos(th)*(sin(th)*cos(phi)^2 + sin(phi)*(cos(th)*sin(ga) + cos(ga)*sin(phi)*sin(th)))]
[ cos(phi)*(sin(ga)*sin(th) - cos(ga)*cos(th)*sin(phi)) + cos(phi)*cos(th)*sin(phi), cos(th)*(cos(ga)*sin(th) + cos(th)*sin(ga)↙
*sin(phi)) - sin(th)*(cos(phi)^2*cos(th) - sin(phi)*(sin(ga)*sin(th) - cos(ga)*cos(th)*sin(phi))), cos(th)*(cos(phi)^2*cos(th) -↙
sin(phi)*(sin(ga)*sin(th) - cos(ga)*cos(th)*sin(phi))) + sin(th)*(cos(ga)*sin(th) + cos(th)*sin(ga)*sin(phi))]

EDU>>
```

P. 2 2 7

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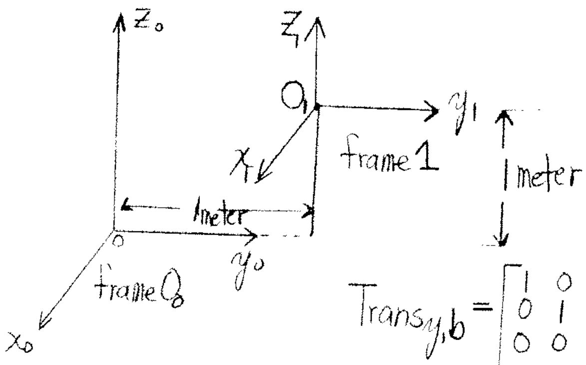
Problem 2-40

Solution:

- Homogeneous transformation from frame 0 to frame 1



Frame 1 is constructed by translating frame 0 by 1-meter in the positive-y direction, and by 1-meter in the positive-z direction, where translation along the y-axis is defined by

$$Trans_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and translation along the z-axis is defined by

$$Trans_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So we can write $H_1^0 = Trans_{y,1} * Trans_{z,1} = \left[\begin{array}{c|c} I & \begin{smallmatrix}0\\1\\0\end{smallmatrix} \\ \hline 0 & 1 \end{array}\right] * \left[\begin{array}{c|c} I & \begin{smallmatrix}0\\0\\1\end{smallmatrix} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} I & I\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right]+\left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right] \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} I & \left[\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right] \\ \hline 0 & 1 \end{array}\right]$

So the coordinate transformation from frame 0 to frame 1 is

$$H_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This transformation represents a vector from the origin O_0 to the origin O_1 expressed in the frame $O_0 x_0 y_0 z_0$.

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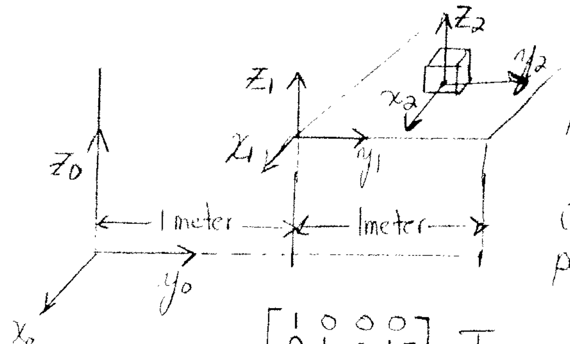
Problem 2-40

Solution:

- Homogeneous transformation from frame 0 to frame 2



Frame 2 is constructed by translating frame 0 by 1.5 meters in the positive y-direction, by 0.5 meters in the negative x-direction, and by 1.1 m in the positive z direction since we are using the center of the cube - which has dimensions of 20 cm per side - as the origin for frame 2.

$$\mathrm{Trans}_{y,1.5} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathrm{Trans}_{x,-0.5} = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathrm{Trans}_{z,1.1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = \mathrm{Trans}_{y,1.5} * \mathrm{Trans}_{x,-0.5} * \mathrm{Trans}_{z,1.1}$$

$$= \left[\begin{array}{c|c} I & \begin{bmatrix} 0 \\ 1.5 \\ 0 \end{bmatrix} \\ \hline 0 & 1 \end{array}\right] * \left[\begin{array}{c|c} I & \begin{bmatrix} -0.5 \\ 0 \\ 0 \end{bmatrix} \\ \hline 0 & 1 \end{array}\right] * \left[\begin{array}{c|c} I & \begin{bmatrix} 0 \\ 0 \\ 1.1 \end{bmatrix} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} I & \begin{bmatrix} 0 \\ 1.5 \\ 0 \end{bmatrix} + \begin{bmatrix} -0.5 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1.1 \end{bmatrix} \\ \hline 0 & 1 \end{array}\right]$$

$$H_2^0 = \left[\begin{array}{c|c} I & \begin{bmatrix} -0.5 \\ 1.5 \\ 1.1 \end{bmatrix} \\ 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This transformation represents a vector from the origin O_0 to the origin O_2 expressed in the frame $O_0 x_0 y_0 z_0$.

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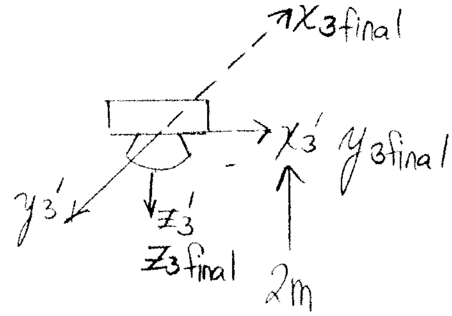
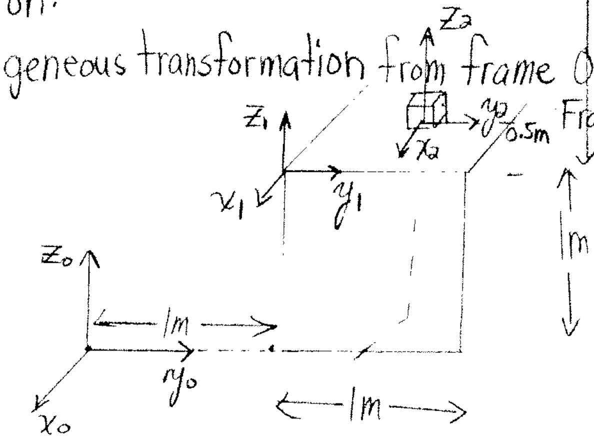


Problem 2-40
Solution:
- Homogeneous transformation from frame 0 to frame 3



Frame 3 is constructed by the following movements:
- Translation along positive-y direction by 1.5 meter
- Translation along negative-x direction by 0.5 meters
- Translation along positive-z direction by 3 meters.
- Rotation along z-direction by −90 degrees.
- Rotation along y-direction by +180 degrees.
- Rotation along z-direction by +90 degrees.

$$H_3^0 = \mathrm{Trans}_{y,1.5} * \mathrm{Trans}_{x,-0.5} * \mathrm{Trans}_{z,3} * \mathrm{Rot}_{z,-90} * \mathrm{Rot}_{y,180} * \mathrm{Rot}_{z,+90}$$

$$= \left[\begin{array}{c|c} I & \begin{bmatrix}-0.5\\1.5\\3\end{bmatrix} \\ \hline 0 & 1\end{array}\right] * \begin{bmatrix} c(-\pi/2) & -s(-\pi/2) & 0 & 0 \\ s(-\pi/2) & c(\pi/2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} * \begin{bmatrix} c(\pi) & 0 & s(\pi) & 0 \\ 0 & 1 & 0 & 0 \\ -s(\pi) & 0 & c(\pi) & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} * \begin{bmatrix} c(\pi/2) & -s(\pi/2) & 0 & 0 \\ s(\pi/2) & c(\pi/2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$= \left[\begin{array}{c|c} I & \begin{bmatrix}-0.5\\1.5\\3\end{bmatrix} \\ \hline 0 & 1\end{array}\right] * \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} * \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} * \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$= \left[\begin{array}{c|c} I & \begin{bmatrix}-0.5\\1.5\\3\end{bmatrix} \\ \hline 0 & 1\end{array}\right] * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & -1 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1\end{bmatrix}$$

By definition, $H_3^0 = H_2^0 * H_3^2 \Longrightarrow H_3^2 = (H_2^0)^{-1} H_3^0$, therefore

$$H_3^2 = \left(\left[\begin{array}{c|c} I & \begin{bmatrix}-0.5\\1.5\\1.1\end{bmatrix} \\ \hline 0 & 1\end{array}\right]\right)^{-1} \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & -1 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1\end{bmatrix} = \left[\begin{array}{c|c} I^T & -I^T\begin{bmatrix}-0.5\\1.5\\1.1\end{bmatrix} \\ \hline 0 & 1\end{array}\right] * \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & -1 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$H_3^2 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \begin{bmatrix}-0.5\\1.5\\3\end{bmatrix} - \begin{bmatrix}-0.5\\1.5\\1.1\end{bmatrix} \\ 0 & -1 & 0 & \\ 0 & 0 & -1 & \\ \hline 0 & 0 & 0 & 1\end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1\end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \begin{bmatrix}0\\0\\1.9\end{bmatrix} \\ 0 & -1 & 0 & \\ 0 & 0 & -1 & \\ \hline & 0 & & 1\end{array}\right]$$