1: Consider the RPR robot used for HW3 (Fig. 1), with the DH parameters of Table 1, where \( d > 0 \).

Point \( P \) has coordinates \( P^3 = [l 0 0]^T \), where \( l > 0 \) is assumed. Consider the following:

- Link 1 has moment of inertia \( I_{z1} \) about the \( z_0 \) axis
- Link 2 has mass \( m_2 \) and center of mass at \( o_2 \)
- Link 3 has a moment of inertia matrix \( I = \text{diag} [I_{x3}, I_{y3}, I_{z3}] \) relative to a frame parallel to frame 3 but centered at the center of mass (halfway between \( o_2 \) and \( o_3 \)). The mass of the link is \( m_3 \).

![Figure 1: 3-dof robot](image)

<table>
<thead>
<tr>
<th>Link</th>
<th>( \theta )</th>
<th>( d )</th>
<th>( \alpha )</th>
<th>( a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( q_1^* )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>( q_2^* )</td>
<td>(-\frac{\pi}{2})</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( q_3^* )</td>
<td>( d )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: DH parameters for RPR robot
Do the following:

1. Find the mass matrix \( M(q) \), Coriolis matrix \( C(q, \dot{q}) \) and gravity vector \( g(q) \) for this robot using symbolic computer algebra. Write clearly commented code and report on the results.

2. Use symbolic computing to verify that \( M \) is symmetric and \( \dot{M} - 2C \) is skew-symmetric.

3. Find a minimal linear parameter representation \( Y(q, \dot{q}, \ddot{q})\Theta \) for this robot. Verify the results as done in class.

2:

- Let \( x \) be a column vector and let \( f(x) = (Ax + b)^T C(Dx + e) \), where \( A, C \) and \( D \) are matrices and \( b \) and \( e \) vectors of compatible dimensions. A formula\(^1\) for the differential of \( f \) with respect to \( x \) is
  \[
  df(x) = ((Ax + b)^T CD + (Dx + e)^T C^T A)dx
  \]
  Assume that \( x \) has 2 elements and that all matrices are 2-by-2. Verify the formula by using symbolic computer algebra (find \( f(x) \) and differentiate explicitly).

- Let \( X \) be a matrix and let \( f(X) = X^{-1} \). A formula for the differential of \( f \) with respect to \( X \) is
  \[
  d(X^{-1}) = -X^{-1}dXX^{-1}
  \]
  Assume that \( X \) is 2-by-2 and verify the formula as above.

3: Consider the multi-input nonlinear control system
  \[
  \dot{x} = A(x)x + (1 + x^T x)Bu
  \]
  where \( A(x) \) is \( n \)-by-\( n \) and \( B \) is \( n \)-by-\( m \) and constant.

- Use the quadratic Lyapunov function
  \[
  V(x) = x^T Px
  \]
  with \( P = P^T > 0 \) to show that the control law
  \[
  u = -(PB)^*(\Gamma + PA(x) + A^T(x)P)x
  \]
  with \( \Gamma = \Gamma^T > 0 \) results in asymptotic stability of the origin, where * denotes the Moore-Penrose pseudoinverse and \( PB \) is assumed full-rank.

- Take
  \[
  A = \begin{bmatrix}
  \sin(x_1)\cos(x_2) & x_1 - x_2 \\
  0 & x_1^2
  \end{bmatrix}, \quad B = \begin{bmatrix}
  0 & 1 \\
  1 & -1
  \end{bmatrix}
  \]
  Select \( P = P^T > 0 \) such that \( PB \) is full-rank and simulate the control system using initial condition \( x^T_0 = [1 \ -2] \). Plot the time histories of each state, the control and a state-space trajectory \((x_1 \text{ vs. } x_2)\). Also plot the Lyapunov function value and its derivative as a function of time.

\(^1\)Matrix Reference Manual, Imperial College: [http://www.ee.ic.ac.uk/hp/staff/dmb/matrix/calculus.html](http://www.ee.ic.ac.uk/hp/staff/dmb/matrix/calculus.html)