Lecture 10: Euler-Lagrange Modeling of Robotic Manipulators

Reading: SHV Ch.7

Mechanical Engineering
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Formulating the Kinetic Energy

The kinetic energy of a rigid body undergoing general motion (translation + rotation) can be expressed as the sum of two components:

Energy due to velocity of the center of mass: This component is simply

$$\frac{1}{2} m v^T v$$

where m is the mass of the body and v is the velocity of the center of mass relative to the world frame.

Energy due to rotation about the center of mass: This component can be expressed as

$$\frac{1}{2} w^T \mathcal{I} w$$

where w is the angular velocity referred to the world frame and $\mathcal{I}$ is a geometry, mass and configuration-dependent term called the *inertia tensor*.

Kinetic Energy...

To compute v, we rigidly attach a coordinate frame to the center of mass. Then we find the Jacobian. This also allows us to find w. If R is the transformation frame between the body-attached frame and the world frame, and I is the inertia tensor expressed in the body-attached frame then

$$\mathcal{I} = RIR^T$$

Note that I depends only on the geometry and mass distribution of the body (independent of q). Tables exist for various shapes.
The inertia tensor in the body frame is defined as

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Kinetic Energy...

The components of the inertia tensor are

$$
\begin{aligned}
I_{xx} &= \int\int\int (y^2 + x^2)\rho(x,y,z)dxdydz \\
I_{yy} &= \int\int\int (x^2 + z^2)\rho(x,y,z)dxdydz \\
I_{zz} &= \int\int\int (x^2 + y^2)\rho(x,y,z)dxdydz \\
I_{xy} = I_{yx} &= -\int\int\int xy\rho(x,y,z)dxdydz \\
I_{xz} = I_{zx} &= -\int\int\int xz\rho(x,y,z)dxdydz \\
I_{yz} = I_{zy} &= -\int\int\int yz\rho(x,y,z)dxdydz
\end{aligned}
$$

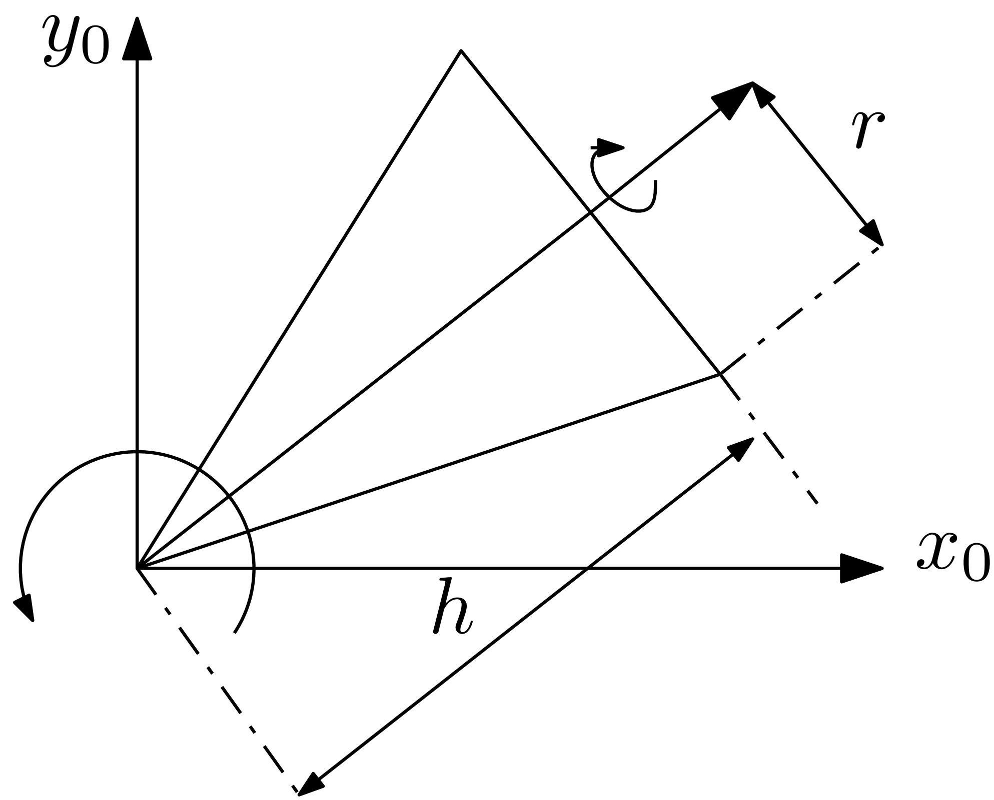
where $\rho(x, y, z)$ is the density distribution (mass per unit volume).

Kinetic Energy...

The terms I_{xx}, I_{yy} and I_{zz} are called *principal moments of inertia*. The other terms are called *products of inertia*. When the body has uniform density (ρ constant) and symmetry about one of the axes, the product of inertia involving that axis is zero.

Example:

Find the kinetic energy of a cone with uniform density which rotates as shown in the figure. Use the methods developed in this course.



Kinetic Energy of a Manipulator

Remember that the velocity of the center of mass and the angular velocity of link i can be obtained by using

$$v_i = J_{v_i}(q)\dot{q} \text{ and } w_i = J_{w_i}(q)\dot{q}$$

The upper Jacobian J_{v_i} must be computed by replacing o_n in Eq. (4.57) by r_{ci}, the position vector of the center of mass of link i.
As derived in SHV, the kinetic energy is found as

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

where $D(q)$ is the *inertia matrix*:

$$D(q) = \left[\sum_{i=1}^{n} m_i J_{v_i}(q)^T J_{v_i}(q) + J_{w_i}(q)^T R_i(q) I_i R_i(q)^T J_{w_i}(q)\right]$$

The inertia matrix $D(q)$ is symmetric and positive-definite for all q

Potential Energy of a Manipulator

The potential energy of the i^{th} link is obtained as

$$P_i = m_i g^T r_{ci}$$

where g is the vector specifying the acceleration of gravity in the world frame, m_i is the mass of the link and r_{ci} is the position vector of the center of mass.

The total potential energy is just the sum over all links:

$$P = \sum_{i=1}^{n} P_i$$

The above applies to rigid links. When links are elastic, the elastic potential energy needs to be included.

Equations of Motion

As derived in SHV, application of the Euler-Lagrange equations to each link results in a system of coupled differential equations. In matrix form:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

Here $D(q)$ is the inertia matrix introduced earlier, $g(q)$ is the gradient of the potential energy ($g_k = \frac{\partial P}{\partial q_k}$) and τ is the vector of joint inputs (forces or torques). The elements of matrix $C(q)$ are defined by

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}\dot{q}_i$$

where c_{ijk} are the *Christoffel Symbols* defined as

$$c_{ijk} = \frac{1}{2}\left\{ \frac{\partial d_{kj}}{\partial q_j} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\}$$

The quantities d_{ij} are the entries of the inertia matrix $D(q)$. Note that for a fixed k,$c_{ijk} = c_{jik}$ (no need to compute all of the c_{ijk}.)

Example

Carefully follow the derivation of the model for the planar elbow manipulator (p.259). In HW5 you will obtain the model for the three-link case.

Model Properties: Skew Symmetry and Passivity

The matrix $N(q, \dot{q}) = \dot{D}(q) - 2C(q, \dot{q})$ is skew-symmetric. This property will be useful when deriving certain control laws.

The *passivity property* states that the mechanical work spent when using joint forces/torques τ to produce joint velocities $\dot{q}$ is lower-bounded by (cannot be less than) certain number $-\beta$:

$$\int_0^T \dot{q}^T(\zeta)\tau(\zeta)d\zeta \geq -\beta$$

with $\beta \geq 0$ and an arbitraty time $T > 0$. Physically, a passive system only dissipates energy. In fact, in a manipulator $\dot{q}^T\tau = \dot{H}$ is power, where H is the energy. Integrating we see that

$$\int_0^T \dot{q}^T(\zeta)\tau(\zeta)d\zeta = H(T) - H(0) \geq -H(0)$$

Model Properties: Bounds on Inertia Matrix

For any frozen configuration q, the inertia matrix $D(q)$ has n positive eigenvalues $0 < \lambda_1(q) \leq \lambda_2(q) \leq ... \leq \lambda_n(q)$. It is possible to show (we do it now) that

$$\lambda_1(q)I \leq D(q) \leq \lambda_n(q)I$$

Show this by diagonalizing $D(q)$ and using the fact that similarity transformations preserve eigenvalues (and therefore sign-definiteness). When the manipulator contains only revolute joints, $D(q)$ contains only sines and cosines, which are bounded by ± 1. Then $D(q)$ can be bounded by explicit constants: $\lambda_m I \leq D(q) \leq \lambda_M I$

Model Properties: Linearity in the Parameters

The dynamic equations of an n-link manipulator can be written as

$$Y(q, \dot{q}, \ddot{q})\Theta = \tau$$

where $Y(q, \dot{q}, \ddot{q})$ is an n-by-l matrix function called the *regressor* and Θ is an $l-$vector of parameters. The minimum number l of parameters is difficult to establish, as it is to find Y.
For many common configurations, the parameterization can be looked up in books/research literature. This property is crucial for advanced schemes like adaptive control and passivity-based control.