Lecture 10.5: LaSalle's Invariance Principle and Barbălat's Lemma

Reading: SHV Appendix, any book on nonlinear control

Mechanical Engineering

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Notes

- The LaSalle-Krasovskii's theorem (known as LaSalle's invariance principle, 1959-1960) and Barbalat's lemma (1959) are useful results that assist in proving asymptotic stability when the Lyapunov function derivative is only negative semi-definite.
- Joseph LaSalle was an American mathematician and Brown University professor.
- I. Barbălat was a Romanian mathematician who published in French. His result is used very often in adaptive control.

LaSalle's Invariance Principle

Let a nonlinear system be defined by

$$\dot{x} = f(x)$$

and suppose the origin is an equilibrium point ($f(0) = 0$).

A set $\mathcal{I} \subseteq \mathcal{R}^n$ is *invariant* if the following is true:

$$x(t_0) \in \mathcal{I} \rightarrow x(t) \in \mathcal{I} \; \forall t > t_0$$

Suppose we found a Lyapunov function $V(x)$ which is positive-definite in a set $\mathcal{D}$ containing the origin, and that $\dot{V}(x) \leq 0$ in $\mathcal{D}$. Consider a set of all trajectories (solutions of the differential equation) that keep $\dot{V} = 0$:

$$\mathcal{E} = \{x \; : \; \dot{V}(x) = 0\}$$

Then all trajectories approach the largest invariant set $\mathcal{I} = \{0\}$ contained in $\mathcal{E}$.

Invariance Principle...

To show asymptotic stability of the origin using LaSalle's result, we first identify $\mathcal{E}$. Then we look for invariant sets under the condition $\dot{V} = 0$. If the only such set $\mathcal{I}$ is the origin itself, we have shown that it is asymptotically stable.

Classical example: pendulum with viscous damping. We do this in class.

Barbălat's Lemma

A function $f : \mathbb{R}^n \to \mathcal{R}$ is *square integrable* if:

$$\int_0^\infty f^T(t) Q f(t) dt \leq \infty$$

A function $g : \mathbb{R}^n \to \mathcal{R}$ is *uniformly continuous* if for any $\epsilon > 0$ there is a $\delta > 0$ such that:

$$||x - y|| < \delta \to ||g(x) - g(y)|| < \epsilon$$

for all $x, y \in \mathbb{R}^n$.

Note that δ can be a function of ϵ but not on the points x and y (that would be plain continuity for g; uniform continuity is a stronger property).

Barbălat's lemma: If $f(t)$ is square integrable and $\frac{df(t)}{dt}$ is uniformly continuous then $\frac{df(t)}{dt} \to 0$ as $t \to \infty$.

Barbălat's Lemma in Lyapunov Theory

Let a nonlinear system be defined by

$$\dot{x} = f(x, \psi(t))$$

where $\psi(t)$ is some input function. Suppose the origin is an equilibrium point ($f(0, \psi) = 0$) for any ψ.

Suppose we found a function $V(x, t)$ which is lower-bounded in a set $\mathcal{D}$ containing the origin, and that $\dot{V}(x, t) \leq 0$ in $\mathcal{D}$. If $\dot{V}(x, t)$ is uniformly continuous with respect to time, then $\dot{V}(x, t) \to 0$ as $t \to \infty$.

Note that uniform continuity of V can be satisfied by verifying that $\ddot{V}$ is bounded.

Barbălat's Lemma Alternatives

Some alternatives have surfaced over the years. For instance, Tao, G. "A simple alternative to the Barbălat Lemma", IEEE Trans. Aut. Ctrl. V42,N5., 1997.

If f is square integrable and has a bounded derivative, then f itself converges to zero asymptotically.

This version has been used in the course notes to prove asymptotic tracking with the adaptive inverse dynamics controller.