Lecture 10.5: LaSalle’s Invariance Principle and Barbălat’s Lemma

Reading: SHV Appendix, any book on nonlinear control

Mechanical Engineering
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The LaSalle-Krasovskii’s theorem (known as LaSalle’s invariance principle, 1959-1960) and Barbalat’s lemma (1959) are useful results that assist in proving asymptotic stability when the Lyapunov function derivative is only negative semi-definite.

Joseph LaSalle was an American mathematician and Brown University professor.

I. Barbălat was a Romanian mathematician who published in French. His result is used very often in adaptive control.
LaSalle’s Invariance Principle

Let a nonlinear system be defined by

\[ \dot{x} = f(x) \]

and suppose the origin is an equilibrium point \((f(0) = 0)\).

A set \( \mathcal{I} \subseteq \mathbb{R}^n \) is *invariant* if the following is true:

\[ x(t_0) \in \mathcal{I} \rightarrow x(t) \in \mathcal{I} \ \forall \ t > t_0 \]

Suppose we found a Lyapunov function \( V(x) \) which is positive-definite in a set \( \mathcal{D} \) containing the origin, and that \( \dot{V}(x) \leq 0 \) in \( \mathcal{D} \). Consider a set of all trajectories (solutions of the differential equation) that keep \( \dot{V} = 0 \):

\[ \mathcal{E} = \{ x : \dot{V}(x) = 0 \} \]

Then all trajectories approach the largest invariant set \( \mathcal{I} = \{0\} \) contained in \( \mathcal{E} \).
To show asymptotic stability of the origin using LaSalle’s result, we first identify $\mathcal{E}$. Then we look for invariant sets under the condition $\dot{V} = 0$. If the only such set $\mathcal{I}$ is the origin itself, we have shown that it is asymptotically stable.

Classical example: pendulum with viscous damping. We do this in class.
Barbălat’s Lemma

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is square integrable if:

\[
\int_0^\infty f^T(t)Qf(t)dt \leq \infty
\]

A function \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) is uniformly continuous if for any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that:

\[
||x - y|| < \delta \rightarrow ||g(x) - g(y)|| < \epsilon
\]

for all \( x, y \in \mathbb{R}^n \).

Note that \( \delta \) can be a function of \( \epsilon \) but not on the points \( x \) and \( y \) (that would be plain continuity for \( g \); uniform continuity is a stronger property).

Barbălat’s lemma: If \( f(t) \) is square integrable and \( \frac{df(t)}{dt} \) is uniformly continuous then \( \frac{df(t)}{dt} \rightarrow 0 \) as \( t \rightarrow \infty \).
Barbălat’s Lemma in Lyapunov Theory

Let a nonlinear system be defined by

$$\dot{x} = f(x, \psi(t))$$

where $\psi(t)$ is some input function. Suppose the origin is an equilibrium point $(f(0, \psi) = 0)$ for any $\psi$.

Suppose we found a function $V(x, t)$ which is lower-bounded in a set $\mathcal{D}$ containing the origin, and that $\dot{V}(x, t) \leq 0$ in $\mathcal{D}$. If $\dot{V}(x, t)$ is uniformly continuous with respect to time, then $\dot{V}(x, t) \to 0$ as $t \to \infty$.

Note that uniform continuity of $V$ can be satisfied by verifying that $\ddot{V}$ is bounded.
Barbălat’s Lemma Alternatives


If \( f \) is square integrable and has a bounded derivative, then \( f \) itself converges to zero asymptotically.

This version has been used in the course notes to prove asymptotic tracking with the adaptive inverse dynamics controller.