# MCE/EEC 647/747: Robot Dynamics and Control

## Lecture 3: Forward and Inverse Kinematics

## Denavit-Hartenberg Convention

Reading: SHV Chapter 3


Mechanical Engineering

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# Aims of Lecture 3

Our goals in this chapter are:

- To be able to find the link-wise and overall homogeneous transformation matrices for a variety of manipulators with 1 dof joints.
- To understand and apply the simplifications introduced by the Denavit-Hartenberg convention.
- To gain understanding of the solution strategies available for the inverse kinematics problem and their use in the open-loop control (non-servo) of manipulators.
- This chapter is a brief and quick overview, enough understand the following topics, which are more relevant to this course.

# Some Important Assumptions

- A manipulator is made up of links and joints (kinematic chain). The links are rigid and the joints are 1 dof, regardless of type (revolute or prismatic). We can always accomodate multi-dof (universal-type) joints by thinking that a link of length 0 sits between degrees of freedom ($d = 0$ in $H$).
- There are $n + 1$ links, numbered from 0 to $n$, 0 being the base (assumed fixed in this course), and $n$ being the end effector.
- There are $n$ joints, numbered from 1 to $n$. Joint $i$ connects link $i - 1$ to link $i$.
- *When joint $i$ is actuated, link $i$ moves*. Consistently with the numbering, nothing moves link 0.
- To each joint there corresponds a single *joint variable*, $q_i$, which is either the angle of rotation (revolute joint) or the displacement (prismatic joint).



# The Transformation Matrix

The transformation matrix between frames (think joints) $i$ and $j$ is denoted $T_j^i$ (to go from $j$ to $i$ according to $p^i = T_j^i p^j$). When $i < j$, this matrix corresponds to a composition of individual transformations:

$$T_j^i = A_{i+1} A_{i+2} ... A_{j-1} A_j$$

Are the $A$'s taken w.r.t. current or fixed frames?
Each $A_i$ is a *variable* matrix, due to link motion. However, each is a function of a single parameter, that is, $A_i = A_i(q_i)$.
We are particularly interested in $T_n^0$, called simply $H$ in SHV.

$$H = T_n^0 = A_1(q_1) A_2(q_2) ... A_n(q_n)$$

Note that the superscript is dropped, since we know that the $A_i$ connects consecutive frames.

# The Transformation Matrix...

Each $A_i$ is a homogeneous matrix:

$$A_i = \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ 0 & 1 \end{bmatrix}$$

The vector $o_i^{i-1}$ is, of course, the one starting at $o_{i-1}$ and ending in $o_i$, with components expressed in frame $i-1$ coordinates.

Note Eq. (3.7) in SHV. It says that instead of multiplying the $A$'s, we can multiply the rotational parts and also use Eq.(3.9) to find a "total" $o_j^i$.



# The Denavit-Hartenberg Convention

A single $A_i$ homogeneous matrix contains, in general, 6 independent parameters (3 rotational dof and 3 translational dof). The D-H convention establishes 2 additional constraints on the relative orientation of consecutive frames (suppose frame 0 and frame 1):

1. The axis $x_1$ is perpendicular to the axis $z_0$ (DH1)
2. The axis $x_1$ intersects the axis $z_0$ (DH2)

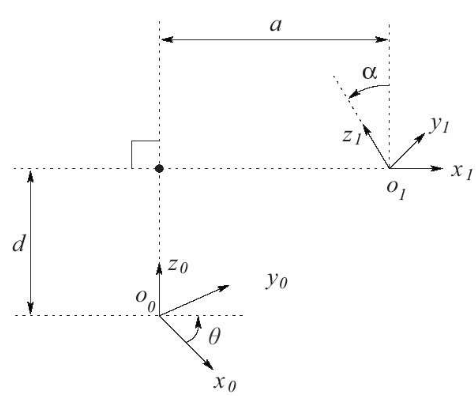

Figure 3.2: Coordinate frames satisfying assumptions DH1 and DH2.

# The D-H Convention...

With DH1 and DH2, the number of free parameters is reduced to 4. Any individual homogeneous transformation $A_i$ can now be resolved as the product of 4 actions:

$$A_i = \text{Rot}_{z,\theta_i}\text{Trans}_{z,d_i}\text{Trans}_{x,a_i}\text{Rot}_{x,\alpha_i}$$

See Eq. (3.10) for the entries of each matrix and the resulting multiplication. The 4 parameters receive certain names:

- $a_i$ is the *link length*
- $\alpha_i$ is the *link twist*
- $d_i$ is the *link offset*, and
- $\theta_i$ is the *joint angle*

Note that only one is variable ($q_i$). The rest are constant and depend on arm geometry.



# Assigning Coordinate Frames

- Sign convention: See Fig.3.3. A positive angle must result in a positive advance along the rotation axis when using the right-hand rule.
- Attach the axes $z_0$, $z_1$...$z_{n-1}$ to each joint, directed along the axis of actuation. Note that $z_i$ is the axis of actuation of joint $i+1$ (reflect on numbering system).
- Establish the remaining axes of the base frame ($x_0$ and $y_0$) arbitrarily, but respecting right-hand rule.
- We define frame $i$ using frame $i-1$. Three cases appear.

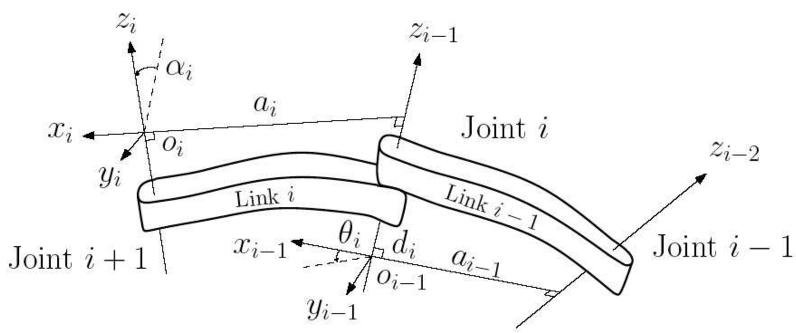


Figure 3.4: Denavit-Hartenberg frame assignment.

# Summary of Coord. Frame Assignment

Please read Sect. 3.2.2. on your own.

- If $z_{i-1}$ and $z_i$ cross without intersecting and are not parallel (not coplanar), find the shortest line which intersects both (this line is unique). Let this line become $x_i$, with $o_i$ being the intersection of $x_i$ with $z_i$. Choose any positive sense for $x_i$ and complete with $y_i$ to form a right-handed frame.
- If $z_{i-1}$ and $z_i$ are parallel, choose $o_i$ anywhere on $z_i$. Choose $x_i$ to be the line joining $o_{i-1}$ to $o_i$ (any positive sense). Then complete as before. If $o_i$ is chosen to be the intersection of the normal to $z_{i-1}$ through $o_{i-1}$ we get $d_i = 0$. Note that $\alpha_i = 0$ always in this case.
- If $z_{i-1}$ and $z_i$ intersect, choose $x_i$ to be the common normal passing through the point of intersection (any positive sense). Usually, $o_i$ is taken to be the point of intersection. Note that $a_i = 0$ always in this case.



# End Frame Assignment

- The coordinate system for the end frame is more conveniently assigned according to end effector features
- The origin $o_n$ is placed in the middle of the tool/gripper
- $a$ is the *approach* direction
- $s$ is the *sliding* direction
- $n$ is the *normal* direction

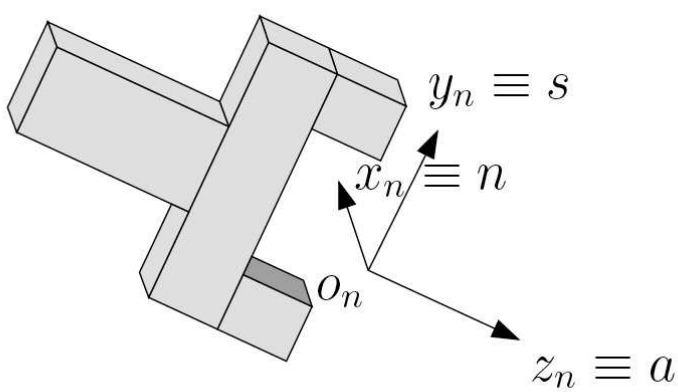


Figure 3.5: Tool frame assignment.

# Examples

- 3.1 In class
- 3.4 In class
- Follow all other examples on your own. Think about alternative solutions.
- Homework 2: 3-10



# Notes

- Forward kinematics: Suppose time histories are given for all joint variables ($q_i(t)$ given). Also, a point $p^n$ is given (for instance the middle of the gripper: $p^n = o_n = 0$). Find the trajectory of $p^n$ in world coordinates (find $p^0(t)$).
- Once $T_n^0$ is determined, all we have to do is substitute the values of $q_i(t)$ and evaluate for all desired time instants.
- Inverse kinematics: The reverse problem ($p^0(t)$ given). We will not cover the details of the solution method.
- Discussion: **Exact model inversions are of little value when considering feedback controls**. The values of the joint variables (control inputs) are not explicit functions of time, but rather function of certain sensed variables. These inputs are determined by the control algorithm, on the basis of the dynamic model (future lectures).