MCE/EEC 647/747: Robot Dynamics and Control

Lecture 4: Velocity Kinematics

Jacobian and Singularities
Torque/Force Relationship
Inverse Velocity Problem

Reading: SHV Chapter 4

Mechanical Engineering

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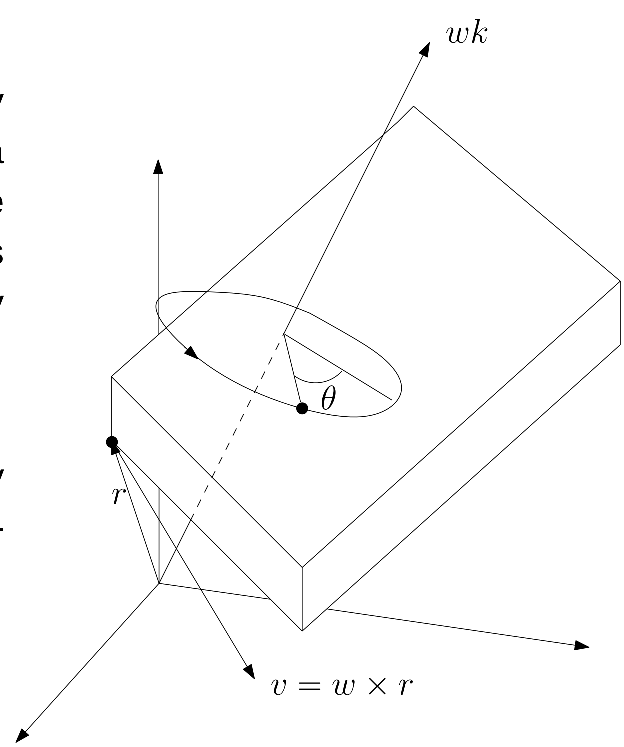
Angular Velocity as a Vector

A point belonging to a rigid body may undergo pure rotational motion about a fixed axis (given by vector k). Then the trajectory of the point is a circle. If $\dot{\theta}$ is the rate of rotation, the angular velocity vector is

$$w = \dot{\theta}k$$

If a point belonging to the same body has position vector r, then its linear velocity is given by

$$v = w \times r$$



Skew Symmetric Matrices

A *skew symmetric* matrix S is defined by the property

$$S + S^T = 0$$

For this to occur, the diagonal elements must be zero and we must have $s_{ij} = -s_{ji}$

The set of all 3-by-3 skew symmetric matrices is called $so(3)$. To each vector $a = [a_x,\, a_y,\, a_z] \in \mathbb{R}^3$ we associate a matrix $S(a) \in so(3)$ of the form

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Properties of Skew Symmetric Matrices

SHV lists 4 properties. The most remarkable ones are 2 and 4:

$$S(a)p = a \times p$$

$$X^T S X = 0$$

Note that *any* matrix A can be decomposed as the sum of a symmetric component and a skew symmetric component. The last property says that the skew symmetric part does not influence a quadratic function defined using A.

Derivative of Rotation Matrices

It is recommended that you familiarize yourself with basic matrix calculus. These results are very useful in many areas of engineering. The Matrix Reference Manual (Imperial College, London) is a very popular document:
`http://www.ee.ic.ac.uk/hp/staff/dmb/matrix/calculus.html`
Following the derivations in SHV we see that

$$\frac{dR}{d\theta} = SR(\theta)$$

where S is a skew symmetric matrix (Eq. 4.13). When $\theta = \theta(t)$, we regard R as a function of time. So

$$\frac{dR}{dt} = S(w)R(t)$$

The w vector can be uniquely extracted from S recalling the earlier association between a skew symmetric matrix and a vector. The w vector is *actually* the angular velocity, as shown in p.125.

Addition of Angular Velocities

Suppose we have a series of coordinate frames with a fixed, common origin. The rotation matrices $R^i_j(t)$ are functions of time, defining the relative orientation of pairs of frames. Let $w^k_{i,j}$ denote the angular velocity associated with the time derivative of R^i_j, with components expressed in the coordinate frame k.
We wish to find the angular velocity of the end effector frame relative to the world frame, expressed in world frame coordinates. That is, we need to find $w^0_{0,n}$. The derivation gives

$$w^0_{0,n} = w^0_{0,1} + R^0_1 w^1_{1,2} + R^0_2 w^2_{2,3} + ... + R^0_{n-1} w^{n-1}_{n-1,n}$$

that is,

$$w^0_{0,n} = w^0_{0,1} + w^0_{1,2} + 0w^0_{2,3} + ... + w^0_{n-1,n}$$

If you think of $w_{i,j}$ as a vector, all we are doing is transforming the various relative angular velocities to the world frame and superimposing them.

Linear Velocity Formula

Suppose point p is rigidly attached to frame 1, which moves (translation + rotation) w.r.t. frame 0, according to the homogeneous transformation

$$P^0 = H_1^0(t)P^1$$

Here $P = [p^T|1]^T$. A simple derivation shows that the linear velocity of p w.r.t. the world frame is

$$\dot{p}^0 = S(w)R_1^0 p^1 + \dot{o} = w \times r + v$$

which is the well-known formula.

Example: We solve Problem 4.15 in class.

The Jacobian

In general, the *Jacobian* of a differentiable function $f : \mathbb{R}^m \mapsto \mathbb{R}^n$ is an n-by-m matrix J where the entries are the first partial derivatives: $J_{ij} = \frac{\partial f_i}{\partial x_j}$ for $i = 1..n$, $j = 1..m$.
A transformation matrix $T_n^0(q)$ is such a function of several variables (exactly equal to the number of joints) returning a vector of dimension 4 (3D case).
In our case, if we form a vector with all joint velocities $\dot{q}$, two Jacobians will be found so that

$$\begin{aligned} v_n^0 &= J_v \dot{q} \\ w_n^0 &= J_w \dot{q} \end{aligned}$$

An overall Jacobian can be formed as $J = [J_v^T|J_w^T]^T$, of dimensions 6-by-n (n is the number of links).

Jacobian...

As derived in SHV, the J_w rows of J are given as $J_w = [\rho_1 z_0 \ \rho_2 z_1 \ ... \ \rho_n z_{n-1}]$ where $\rho_i = 1$ if joint i is revolute and $i = 0$ if it is prismatic. z_i is the axis of rotation of joint i expressed in the world frame coordinates.
The $i-$th column of J_v is given as

$$J_{v_i} = \frac{\partial o_n^0}{\partial q_i}$$

where o_n^0 is the position vector for the origin of the end effector frame, in world coordinates.

Note that the velocity $\dot{o}_n^0$ is obtained by a linear combination of the columns of J_v where the scalars are the joint velocities (Eq. 4.49). Therefore J_v can be obtained by taking one joint at a time (assuming the others are not moving) and assembling the resulting columns into a matrix.

Summary of Formulas for the Jacobian

Please read Section 4.6 in detail.
The upper half of the Jacobian is given as $J_v = [J_{v_1}, ...J_{v_n}]$ where the columns are given by

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for revolute joint } i \\ z_{i-1} & \text{for prismatic joint } i \end{cases}$$

The lower half is given also in columns by

$$J_{v_i} = \begin{cases} z_{i-1} & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases}$$

Examples

Note: The coordinates of z_i w.r.t. the world frame appear as the first three elements of the third column of T_i^0, while o_i appears as the first three elements of the fourth column. Therefore we need to find only the third and fourth columns of T.

- 4.5, 4.6 in class
- Prob. 4.16 in class
- Homework 2: 4-17

Singularities

For a given configuration q, we can view the Jacobian as a linear transformation mapping joint velocities $\dot{q}$ to end-effector velocities $\zeta = [v^T|w^T]^T$. The velocity vector is then a linear combination of the columns of J:

$$\zeta = J_1\dot{q}_1 + J_2\dot{q}_2 + ... + J_n\dot{q}_n$$

that is, $\zeta = J(q)\dot{q}$.
For 3D manipulators ζ is a 6-vector. A condition on $J(q)$ so that a $\dot{q}$ exists that results in any desired ζ is that $J(q)$ be *full rank*. (Please see Appendix B if needed).
When $J(q)$ loses rank for a particular value of q, we encounter a *singularity*.

When the manipulator is in a singular configuration, not all directions of motion are possible for the end effector. We consider the effect of singularities in terms of end effector forces and torques later in the course.

Static Torque/Force Relationship

Suppose the end effector is in static equilibrium and subjected to forces and moments (for example pressing against a part). These forces and moments will create torques at revolute joints and forces at prismatic joints. If we organize the end-effector external forces F_i and moments n_i in a vector $F = [F_x \, F_y \, F_z \, n_x \, n_y \, n_z]$ and the resulting joint torques in a vector τ, the relationship is given by Eq. 4.105 in SHV:

$$\tau = J^T(q)F$$

It is important to note that this relationship is valid only under static equilibrium. See Example 4.12.

Linear Algebra Facts: Moore-Penrose Pseudoinverse

Suppose J is an m-by-n matrix with $m \neq n$. Then the linear system of equations

$$Jx = a$$

may or may not have solutions.

- If $m \neq n$, solutions exist only if vector a is a linear combination of the columns of J (a in the range space or column space of J, usually expressed as $a \in$ col J.
- To check: $a \in$ col J if and only if rank(J)=rank($[J|a]$)
- If $m < n$ and rank(J)=m, JJ^T is an m-by-m invertible matrix. We use this to define

  $$J^+ = J^T(JJ^T)^{-1}$$

  called the *right pseudoinverse of* J, since $JJ^+ = I$.

Pseudoinverse...

As an example, we determine the solvability and find one solution to the system of equations

$$\begin{aligned} x + y - z + w &= 3 \\ x - y + 2z - w &= 8 \\ 2x + y + z + 3w &= 9 \end{aligned}$$

using the Matlab commands `rank` and `pinv`

Inverse Velocity Problem

We wish to determine the vector of joint velocities $\dot{q}$ which results in a desired end-effector velocity ζ. The Jacobian relationship $\zeta = J\dot{q}$ can be inverted to give $\dot{q} = J^{-1}\zeta$ only if J is square and has no singularities. When the robot has more than 6 joints, J is be non-square. If rank J=6, we can use the pseudoinverse to find a solution. Direct substitution shows that

$$\dot{q} = J^{+}\zeta + (I - J^{+}J)b$$

is a solution, with b being an arbitrary vector.
Note that any vector of the form $(I - J^{+}J)b$ is such that

$$J(I - J^{+}J)b = 0$$

which shows that it is possible to have a zero end-effector velocity with non-zero joint velocities!