Lecture 4.5: Manipulability

Reading: SHV Sect.4.12

Mechanical Engineering
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The Singular Value Decomposition

Let $A$ be the matrix of any linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$. We know that $A$ rotates and changes the length of vectors. The singular value decomposition (SVD) explains these geometric transformations completely. Fact:

1. Any $m$-by-$n$ matrix $A$ can be decomposed as

$$A = U \Sigma V^T$$

where $U$ and $V$ are orthogonal and $\Sigma$ has the following structure:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where: } \Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma_p \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_p > 0$ and $p = \min\{m, n\}$. 
Geometric Interpretation

Transformation $u = Au$ takes a vector $v \in \mathbb{R}^n$ and returns a vector $u \in \mathbb{R}^m$. If $v$ is arbitrarily varied under the restriction $||v|| = 1$:

1. The images $u$ describe an ellipsoid.

2. The length of the major axis is $\sigma_1$ and $\sigma_p$ is the length of the minor axis.

3. The right singular vector $v_1$ results in the maximum amplification ($||u_1||/||v_1|| = \sigma_1$). The image $u_1$ is the direction of the ellipsoid’s major axis.

4. The right singular vector $v_n$ results in the least amplification ($||u_2||/||v_2|| = \sigma_p$). The image $u_2$ is the direction of the ellipsoid’s minor axis.

Matlab: $[U, S, V] = \text{svd}(A)$
Example

For an arbitrary 2x2 nonsingular matrix:

1. Use Matlab to vary \(v\) with \(\|v\| = 1\) using the polar coordinate parameterization.

2. Compute and plot the images to visualize the ellipse.

3. Obtain the SVD

4. Calculate \(Av_1/\sigma_1\) to verify \(u_1\) is obtained. Similarly with \(Av_2/\sigma_2\).

5. Plot vectors \(u_1\) and \(u_2\) to verify that they coincide with the ellipse’s principal axes.

These steps are carried out in `exampleSVD.m`. 
Regard \( v \) as the vector of joint velocities, 
\[
\begin{align*}
v^T &= \dot{q}^T = [\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n].
\end{align*}
\]

Think of the Jacobian (at a fixed \( q \)) as the matrix of a linear transformation: 
\[
A = J(q)
\]
with respect to the world frame basis.

The problem is to determine the set of attainable velocity vectors (linear and angular) under a fixed “joint velocity budget”. It’s enough to consider

\[
||\dot{q}|| \leq 1
\]

since constants other than 1 result in a simple scaling.

This is clearly related to the SVD.
Manipulability Ellipsoid

Let the velocity vector (linear and angular) be $\zeta$ (a 6x1 vector). With a full-rank Jacobian, the following is a solution for $\dot{q}$:

$$\dot{q} = J^+ \zeta$$

Then the set of velocities such that $||\dot{q}|| \leq 1$ is given by

$$\zeta^T (J J^T)^{-1} \zeta \leq 1$$

This defines an $m$-dimensional ellipsoid, called *manipulability ellipsoid*
If the SVD of $J = U \Sigma V^T$ is used, it is possible to show that the ellipsoid can be described as

$$w^T \Sigma_m^{-2} w \leq 1$$

where $w = U^T \zeta$ (a coordinate transformation to the ellipsoid’s principal axes, recall that $U$ is a rotation matrix), and

$$\Sigma_m = \text{diag} \left( \sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_m^{-2} \right)$$

This becomes the familiar equation for an ellipsoid:

$$\frac{w_1^2}{\sigma_1^2} + \frac{w_2^2}{\sigma_2^2} + \ldots + \frac{w_m^2}{\sigma_m^2} \leq 1$$

In other words, the singular values of $J$ are the lengths of the ellipse’s axes, and the volume of the ellipsoid (a measure of manipulability) is proportional to $\sigma_1 \sigma_2 \ldots \sigma_m$. 
Manipulability Ellipsoid...

- \( \text{null}(J(q)) \) is formed by the right singular vectors corresponding to zero singular values.

- The set of attainable velocity vectors at a given \( q \) is \( \text{col}(J(q)) \).

Example: Describe the manipulability ellipsoid of the unit length, 2-link planar manipulator at \( q_1 = 0 \) and \( q_2 = \pi/4 \) (linear velocity only). Repeat for \( q_1 = \pi/4 \), \( q_2 = 0 \).
Manipulability Measures

One measure of a robot’s manipulability is given by the ratio of the Jacobian’s maximum to the minimum singular value (the *condition number*).

Matlab computes the condition number with `cond`. The larger the condition number, the closest the matrix is to being singular.

Isotropic manipulability is obtained when the condition number is one (ellipsoid is a sphere).

A measure of the volume of the manipulability ellipsoid was introduced by Yoshikawa as

$$\mu(q) = \sqrt{\det(J^T(q)J(q))}$$

When the Jacobian is square, $\mu(q) = \det(J(q))$. 
Yoshikawa’s Manipulability

\[ \mu(q) = \sqrt{\det(J^T(q)J(q))} \]

This measure is convenient since \( J^TJ \) is always positive semi-definite and provides a measure of the ellipsoid’s volume. Also \( \mu(q) = 0 \) only when the Jacobian is singular.

However, it is a non-convex measure, which can present difficulties for certain optimization problems. A function \( f : X \mapsto \mathbb{R} \) with \( X \in \mathbb{R}^n \) is convex in \( X \) if \( \forall x_1, x_2 \in X \)

\[ f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \]

\( \forall t \in [0, 1] \).

“The image of an interpolation of two points is an interpolation of the images of the points, not an extrapolation.”
Some Optimization Problems

- For a 2-link planar manipulator, prove that Yoshikawa’s manipulability measure and the condition number are both independent of the first joint coordinate. (Doctoral HW).

- For the planar manipulator, plot the manipulability ellipses at a few values of $q_2$ for $q_1 = 0$. Determine the best value of $q_2$ visually.

- Find the best posture (optimal value of $q_2$) for a given planar manipulator per Yoshikawa’s measure by formal maximization. (HW)

- Optimize the geometry of a manipulator to maximize $\mu(q)$ at a given $q$ (potential project)