MCE/EEC 647/747: Robot Dynamics and Control

Lecture 8: Basic Lyapunov Stability Theory

Reading: SHV Appendix

ОБЩАЯ ЗАДАЧА

ОБЪ

СТОЙЧИВОСТИ ДВИЖЕНІЯ.

РАЗСУЖДЕНІЕ

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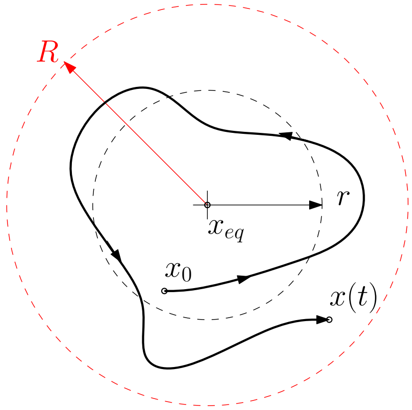
Stability in the sense of Lyapunov

A dynamic system $\dot{x} = f(x)$ is *Lyapunov stable* or *internally stable* about an equilibrium point x_{eq} if state trajectories are confined to a bounded region whenever the initial condition x_0 is chosen sufficiently close to x_{eq}. Mathematically, given $R > 0$ there always exists $r > 0$ so that if $||x_0 - x_{eq}|| < r$, then $||x(t) - x_{eq}|| < R$ for all $t > 0$. As seen in the figure R defines a desired confinement region, while r defines the neighborhood of x_{eq} where x_0 must belong so that $x(t)$ does not exit the confinement region.



Stability in the sense of Lyapunov...

Note:

- Lyapunov stability does not require $||x(t)||$ to converge to $||x_{eq}||$. The stronger definition of *asymptotic stability* requires that $||x(t)|| \rightarrow ||x_{eq}||$ as $t \rightarrow \infty$.
- Input-Output stability (BIBO) does not imply Lyapunov stability. The system can be BIBO stable but have unbounded states that do not cause the output to be unbounded (for example take $x_1(t) \rightarrow \infty$, with $y = Cx = [01]x$).
- The definition is difficult to use to test the stability of a given system. Instead, we use Lyapunov's stability theorem, also called Lyapunov's direct method. This theorem is only a sufficient condition, however. When the test fails, the results are inconclusive. It's still the best tool available to evaluate and ensure the stability of nonlinear systems.

Lyapunov's Linearization Method

This method allows us to determine the stability of the nonlinear system about the equilibrium point on the basis of the linearized system. Simply:

- If the eigenvalues of the A matrix in the linearized system have negative real parts, the nonlinear system is stable about the equilibrium point.
- If at least one eigenvalue of the A matrix in the linearized system has positive real part, the nonlinear system is unstable about the equilibrium point.
- If at least one eigenvalue ofthe A matrix in the linearized system has zero real part, the test is inconclusive. The linear approximation is insufficient to determine stability. However, methods exist to include higher-order terms (center manifold technique).

Positive-Definite Functions and Matrices

The function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is *positive semidefinite* if $f(0) = 0$ and $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. If $f(x) > 0$ for all nonzero $x \in \mathbb{R}^n$, then f is *positive definite*. Example:

$$f(x_1, x_2) = x_1^2 + x_2^2 + \sin^2(x_1)$$

is positive-definite.
If $f(x) \geq 0$ (resp. $f(x) > 0$) in a subset $A \subset \mathbb{R}^n$, we say that f is positive-semidefinite (resp. positive definite) in A.
When $f(x)$ is a quadratic function defined through a symmetric matrix P as

$$f(x) = x^T P x$$

then we say that P is *positive-semidefinite* (resp. positive definite) when the associated function is so.

Finally, if $-f$ is positive-semidefinite (resp. positive definite), we say that f is *negative-semidefinite* (resp. negative definite). If f changes sign, it is *sign-indefinite*.

Testing for Positive-Definite Matrices

Suppose $f(x) = x_1^2 + 2x_2^2 - 0.1x_3^2 - x_1 x_2 + 8x_2 x_3 - 5x_1 x_3$. How can we tell if f is sign-definite?
The symmetric matrix associated with f is

$$P = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & 2 & 4 \\ -\frac{5}{2} & 4 & 0.1 \end{bmatrix} \tag{1}$$

The *eigenvalue* test can be applied. If all eigenvalues are nonnegative, we have a p.s.d. matrix. If they are all positive, we have a p.d. matrix. The same idea applies for n.s.d. and n.d. If the eigenvalues have mixed signs, the matrix is sign-indefinite.

Lyapunov's Direct Method

For simplicity, suppose the origin is an equilibrium point. Suppose we find a function $V : \mathbb{R}^n \mapsto \mathbb{R}$ which has continuous first partial derivatives and is positive-definite in a region surrounding the origin. We call this function a *candidate Lyapunov function*. If the function $\dot{V}$ is negative-semidefinite in the same region, then the origin is a stable equilibrium point.

When the region where V is p.d. and $\dot{V}$ is n.s.d. can be extended to the whole state-space, we say that the origin is *globally stable*.

Necessary vs. Sufficient

Suppose we have a condition A and a property B. (think $V > 0, \dot{V} \leq 0$ as a condition and stability as the property)
We have several possible logical relationships:

- **When A is true, then B must be true, but if A is false, nothing can be said about B.** Then A is a *sufficient condition*, represented as $A \rightarrow B$ (A implies B). Example: If you live in the USA (A), then you live in North America (B). (if you say you don't live in the USA, we can't tell whether you live in North America or not).
- **When B is true, then A must be true, but if B is false, nothing can be said about A.** Then A is a *necessary condition*, represented as $B \rightarrow A$ (B implies A). Example: Living in North America (A) is a necessary condition for living in the USA (B).
- **When A is true, then B must be true, and if A is false, then B is also false**. Then A and B are *necessary and sufficient* conditions for each other, represented as $A \leftrightarrow B$. We also say A *if and only if* B. Example: A linear system is stable if and only if the eigenvalues of A have negative real parts

Sufficient Conditions and Conservativeness

It is important to realize that sufficient conditions are conservative. Suppose you are trying to determine whether someone lives in North America. Then you decide to use the question “Do you live in the USA?” as a test. Your condition is only sufficient. It doesn’t help you when the answer is “No”. As a result, you will not be able to accurately estimate how many people lives in North America from those who you interviewed.

It is important to realize that the Lyapunov theorem is only sufficient. As a consequence, if the Lyapunov function candidate does not have n.s.d. derivative, it does not mean that the system is unstable.

There are some instability theorems, however (sufficient condition for instability). Of course, if they fail, it doesn’t mean the system is stable!

Example

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2 u \\ \dot{x}_2 &= -x_1 u\end{aligned}$$

The origin is an equilibrium point. Check the stability of the system using the Lyapunov function candidate

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$$

Asymptotic Stability and Global Asymptotic Stability

Stronger conditions on V and its derivative are required for asymptotic stability and global asymptotic stability

1. If $\dot{V}$ is negative-definite ($\dot{V} < 0$), then the origin is asymptotically stable.
2. If in addition $V(x) \to \infty$ as $||x|| \to \infty$ (V is *radially unbounded*), then the origin is globally asymptotically stable.

Global asymptotic stability, also known as *asymptotic stability in the large* is the most desirable situation.

Region of Attraction

Suppose an equilibrium point x_{eq} is asymptotically stable, but not globally. We would like to determine the largest region surrounding x_{eq} which results in trajectories which converge to x_{eq}. Such region is called *region of attraction*.

In practical terms, if we design a closed-loop nonlinear control system to regulate the state to x_{eq}, we want to determine the safe region of operation.

A somewhat conservative way to estimate this region is through a Lyapunov function. The estimated region of attraction is the region in which $V > 0$ and $\dot{V} < 0$. When $V(x)$ and $f(x)$ are sufficiently simple, we may determine this region analytically. Otherwise we can use a numerical technique (HW4).

This method is conservative because the Lyapunov stability theorem is only sufficient, not necessary. That is, the actual region of attraction will be larger.

2D and 3D: Region of Attraction by Spherical Sweep

When $\dot{V}$ is complex enough to prevent analysis, we can explore the region where $\dot{V} < 0$ by a numerical method. A simple way to do this is to determine a spherical region. This will introduce additional conservativeness, since the actual region of attraction is generally not spherical. We determine a radius of attraction by setting up a computer iteration (3 nested `for` loops) that vary the r, θ and ϕ parameters of the spherical coordinate system.

For each combination of r, θ and ϕ, we find the corresponding x_1, x_2, x_3 coordinates and evaluate $\dot{V}$ to determine its sign. We continue until $\dot{V} > 0$. The limiting value of r gives us the radius of attraction estimate.

Example

Consider the nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -\sin(x_1)\cos(x_2)\\ \dot{x}_2 &= -\cos(x_1)\sin(x_2)\end{aligned}$$

1. Find the equilibrium points and use Lyapunov's linearization method to determine their stability
2. Use the quadratic Lyapunov function $V(x_1, x_2) = \frac{1}{2}((x_1 - \pi)^2 + (x_2 + \pi)^2)$ to study the stability of the equilibrium point $(\pi, -\pi)$. Determine a region of attraction by a computer-generated contour plot and a radius of attraction by a polar coordinate sweep.

Quadratic Lyapunov Functions and Linear Systems

Consider the linear system $\dot{x} = Ax$ and the quadratic Lyapunov function $V(x) = x^T P^x$, with P being symmetric and positive-definite.
The derivative of V is

$$\dot{V} = \dot{x}^T Px + x^T P\dot{x} = x^T(A^T P + PA)x$$

The linear system defined by A is asymptotically stable (A is Hurwitz) if $A^T P + PA$ is negative-definite. This is referred to as a *Lyapunov inequality*. We can turn this into an equality by making

$$A^T P + PA = -Q$$

where Q is an arbitrary positive-definite matrix. We can solve for P using Matlab's `lyap` command. If P turns out to be positive-definite, we conclude asymptotic stability.

Note: Matlab's `lyap` uses the transpose of A.

Sufficiency of Lyapunov's Theorem for Linear Systems

For linear systems, the existence of P so that $A^T P + PA < 0$ is necessary and sufficient for stability:

- If the linear system is asymptotically stable, there has to be a P proving it. We can find a P by choosing an arbitrary $Q > 0$ and solving Lyapunov's equation.
- If the linear system is unstable and the Lyapunov equation is solved for P using an arbitrary $Q > 0$, P will not be positive-definite or a unique solution may not exist.

In summary, any quadratic function $V = x^T Px$ can be used to prove the asymptotic stability of a linear system.

Example: Linear Systems

Find a quadratic Lyapunov function for the linearized system of the previous example about $(\pi, -\pi)$ using Lyapunov's equation. Use a similar method to prove that $(\pi/2, \pi/2)$ is unstable.