Lecture 9: Introduction to Euler-Lagrange Modeling

Reading: SHV Ch.7

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Lagrangian mechanics

■ Energy-based approach
■ Eliminates forces of constraints and conservative forces from the formulation
■ Kinetic Energy (simple case)
  ♦ Translational: $T = \frac{1}{2}mv^2$
  ♦ Rotational : $T = \frac{1}{2}Iw^2$
■ Potential Energy
  ♦ Gravitational: $U = mgz$
  ♦ Elastic: $U = \frac{1}{2}k(\Delta x)^2$
The Euler-Lagrange equation

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0
\]

- Lagrangian \( L = T - U \)
- \( L = f(q_1, q_2...q_n, \dot{q}_1, \dot{q}_2, ...\dot{q}_n) \)
- \( q_i \) are the \textit{generalized coordinates}
- \( q_i \) can be distances, angles or arbitrary combinations of geometric and physical quantities
- The set of \( q_i, \ i = 1..n \) must completely specify the state of the system.

Validity of E-L equation

NOTES: The E-L equation is valid only if

1. Forces are conservative (they are the gradient of a potential: \( \exists U \ \text{s.t.} \ \vec{F} = -\nabla U \).
2. Forces of constraint are non-dissipative and do no virtual work

Forces arising from gravitation are conservative \( (U = mgz) \)
Forces arising from ideal springs are conservative \( (U = 0.5k(\Delta x)^2) \)
Normal forces and tension satisfy (2)
Friction forces (dry or viscous) do not satisfy (1) or (2), but terms can be added to E-L equation.
Extended E-L equation

- In control applications we have external (non-conservative) forces: control inputs
- Damping is usually present in our models
- Extended equation:
  \[
  \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = F_i
  \]
  - \( q_i \) is the generalized external force (conservative ones can be pulled out of the potential as well)
  - Rayleigh’s dissipation function: \( \mathcal{R}(\dot{q}) = \sum \frac{b_i q_i^2}{2} \)

Simple example: quarter-car suspension

Find the I/O differential equation \((x, y)\)
More difficult example

Find the I/O differential equation \((F_y)\)
Verify that the required equation is

$$\left( M + \frac{I}{r^2} \right) \ddot{y} + b \dot{y} + \left( k_M + k \frac{k}{4} \right) y = \frac{F}{2}$$
Example on E-L equations

Coupled pendulum, small oscillations

* 2 d.o.f.

* Assume eq. position (unstretched spring) is

\[ \theta_1 = \theta_2 = 0 \]