Lecture 9: Introduction to Euler-Lagrange Modeling

Reading: SHV Ch.7

Mechanical Engineering

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Lagrangian mechanics

- Energy-based approach
- Eliminates forces of constraints and conservative forces from the formulation
- Kinetic Energy (simple case)
  - Translational: $T = \frac{1}{2}mv^2$
  - Rotational : $T = \frac{1}{2}Iw^2$
- Potential Energy
  - Gravitational: $U = mgz$
  - Elastic: $U = \frac{1}{2}k(\Delta x)^2$

The Euler-Lagrange equation

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$

- Lagrangian $L = T - U$
- $L = f(q_1, q_2 ... q_n, \dot{q}_1, \dot{q}_2, ... \dot{q}_n)$
- q_i are the *generalized coordinates*
- q_i can be distances, angles or arbitrary combinations of geometric and physical quantities
- The set of $q_i,\ i = 1..n$ must completely specify the state of the system.

Validity of E-L equation

NOTES: The E-L equation is valid only if

1. Forces are conservative (they are the gradient of a potential: $\exists U$ s.t. $\vec{F} = -\nabla U$.
2. Forces of constraint are non-dissipative and do no virtual work

Forces arising from gravitation are conservative ($U = mgz$)
Forces arising from ideal springs are conservative ($U = 0.5k(\Delta x)^2$)
Normal forces and tension satisfy (2)
Friction forces (dry or viscous) do not satisfy (1) or (2), but terms can be added to E-L equation.

Extended E-L equation

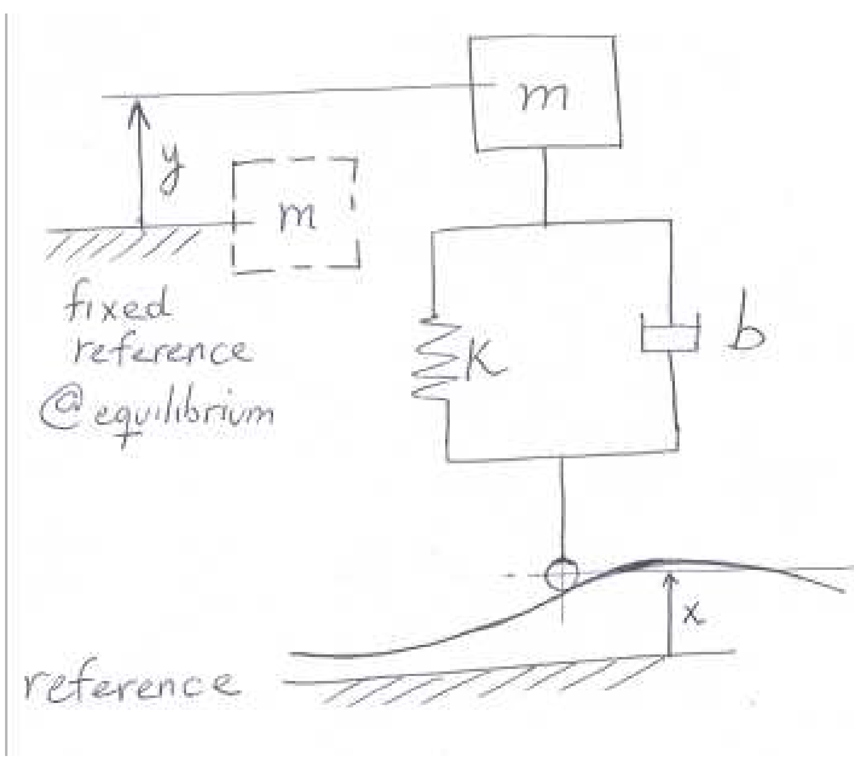
- In control applications we have external (non-conservative) forces: control inputs
- Damping is usually present in our models
- Extended equation:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathcal{R}}{\partial \dot{q}_i} = F_i}$$

  - q_i is the generalized external force (conservative ones can be pulled out of the potential as well)
    - Rayleigh's dissipation function: $\mathcal{R}(\dot{q}) = \sum \frac{b_i \dot{q}_i^2}{2}$

Simple example: quarter-car suspension

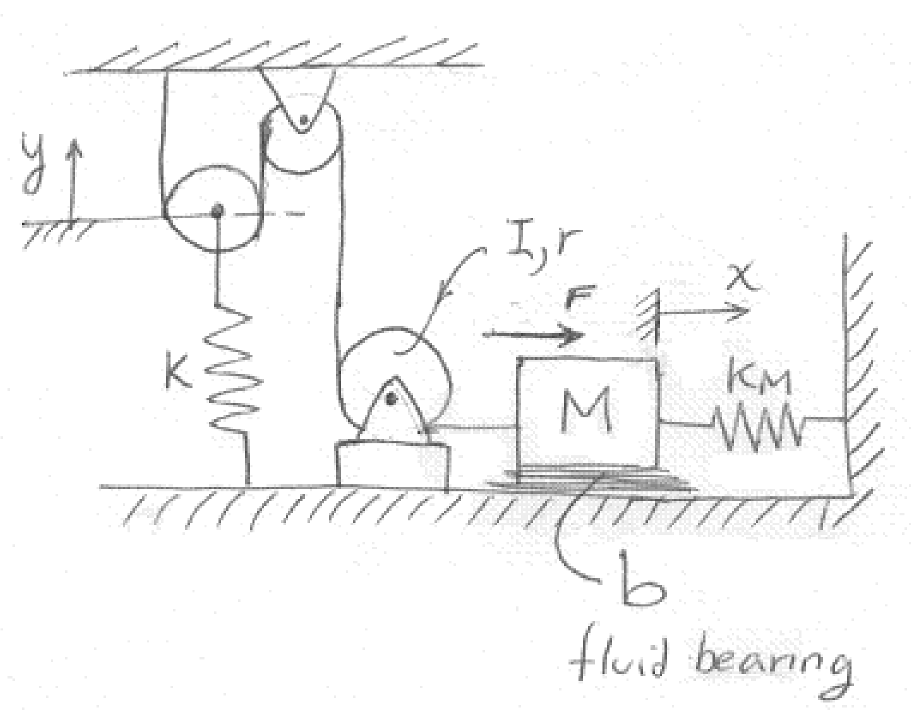
Find the I/O differential equation (x, y)



Solution

More difficult example

Find the I/O differential equation (F,y)



Solution

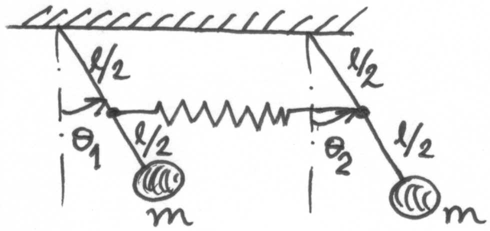
Solution

Verify that the required equation is

$$\left(M + \frac{I}{r^2}\right)\ddot{y} + b\dot{y} + \left(k_M + \frac{k}{4}\right)y = \frac{F}{2}$$

Example on E-L equations

Coupled pendulum, small oscillations



* 2 .d.o.f.
* Assume eq. position (unstretched spring) is $\theta_1 = \theta_2 = 0$