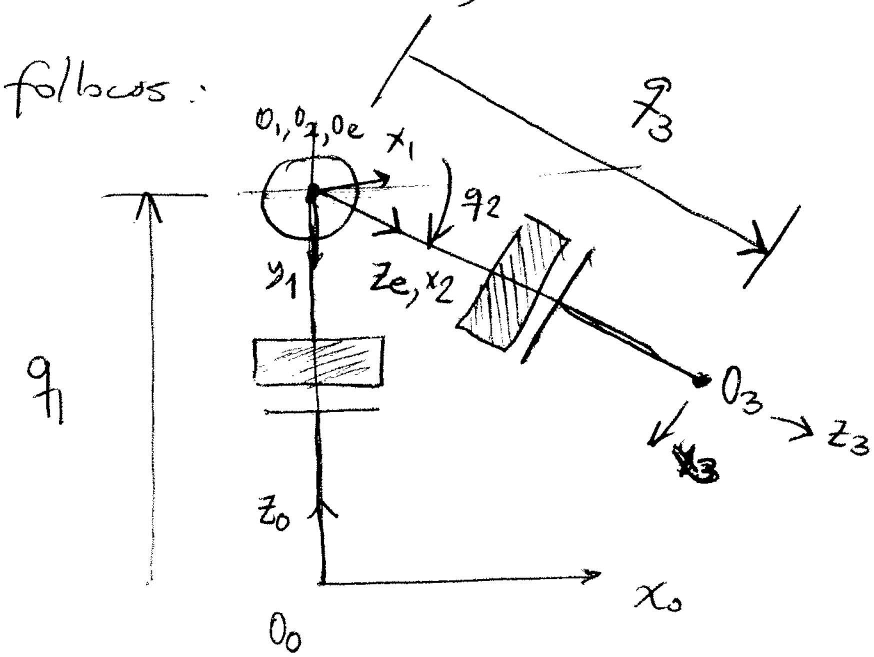
MCE/EEC 647/747 Midterm Solution – Spring '15

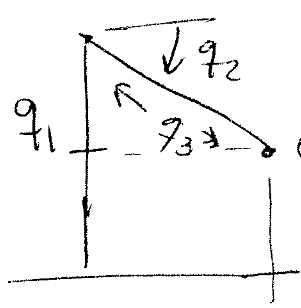
— Dr. Richter —

1. From the DH table, the robot and frames are as follows:




Clearly this is a planar robot (incapable of moving into the $y_0$ direction)

Position of $O_3$:

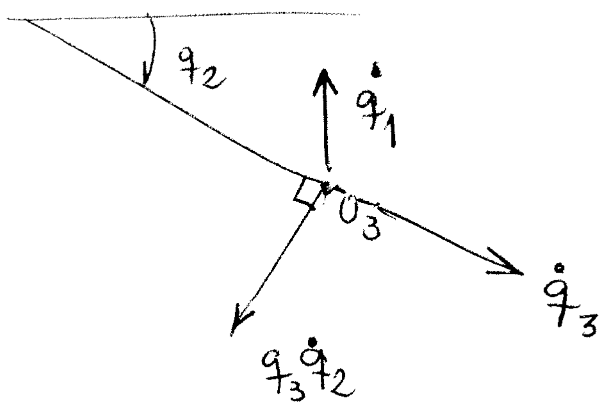



$O_{3y} = q_1 - q_3 \sin q_2$

$O_{3x} = q_3 \cos q_2$

$$O_3^0 = \begin{bmatrix} q_3 \cos q_2 \\ 0 \\ q_1 - q_3 \sin q_2 \end{bmatrix}$$

We can find the world velocity of $O_3$ by vector addition of velocities:




$$v_x = \dot{q}_3 \cos q_2 - \dot{q}_2 q_3 \sin q$$

$$v_y = 0$$

$$v_z = \dot{q}_1 - \dot{q}_3 \sin q_2 - q_3 \dot{q}_2 \cos q_2$$

In vector-matrix form:

$$\dot{O}_3 = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -q_3 s_2 & c_2 \\ 0 & 0 & 0 \\ 1 & -q_3 c_2 & -s_2 \end{bmatrix}}_{\text{Jacobian.}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

(singlar).

The robot can't produce a nonzero velocity in the $y$ direction. $y$ is restricted to 0.

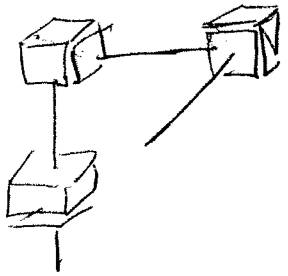
Manipulability ellipsoid: $\eta(q) = \sqrt{\det(J^T J)}$

$$J^T J = \begin{bmatrix} 0 & 0 & 1 \\ -q_3 s_2 & 0 & -q_3 c_2 \\ c_2 & 0 & -s_2 \end{bmatrix} \begin{bmatrix} 0 & -q_3 s_2 & c_2 \\ 0 & 0 & 0 \\ 1 & -q_3 c_2 & -s_2 \end{bmatrix} = \begin{bmatrix} 1 & -q_3 c_2 & -s_2 \\ -q_3 c_2 & q_3^2 & 0 \\ -s_2 & 0 & 1 \end{bmatrix}$$

Because $J$ is square, no need to find $\det(J^T J)$

we know $\eta(q) = |\det(J)| = 0$

2. If the three P joints are orthogonal

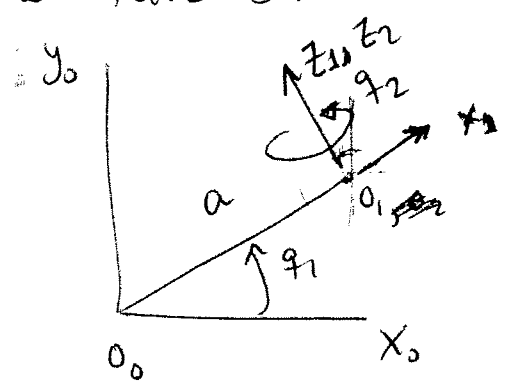


then $v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \underset{\uparrow \\ \text{Jacobian}}{I} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$

Then there cannot be singularities.

If at least 2 joints are parallel, the manipulator is always singular (in 3D).

3. From the D-H table, the robot and frames are as follows:




From this figure:

$$z_1^0 = z_2^0 = \begin{bmatrix} -\sin q_1 \\ \cos q_1 \\ 0 \end{bmatrix}$$

$$z_0^0 = \hat{k}$$

$$O_1^0 = \begin{bmatrix} a \cos q_1 \\ a \sin q_1 \\ 0 \end{bmatrix}$$

To compute the linear velocity Jacobian, we need only

$J_{v_1} = z_0 \times O_2^0$ — point of interest

$J_{v_2} = z_1 \times (O_2^0 - O_1^0)$

So only $O_2^0$ is needed! $\quad O_2^0 = A_2^0\, O_2^2 = A_2^0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$A_1^0 = Rot_{z,q_1} \cdot Rot_{x,-\pi/2} \cdot Trans_{x,a}$$

$$A_1^0 = \left[\begin{array}{ccc|c} \cos q_1 & 0 & -\sin q_1 & a\cos q_1 \\ \sin q_1 & 0 & \cos q_1 & a\sin q_1 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \quad o_1^0$$

find this from the rotations only (one 3x3 multiplication)

$$A_2^1 = Rot_{z,q_2}$$

only these are needed!

$$\text{Multiply: } A_1^0 A_2^1 = A_2^0 = \left[\begin{array}{ccc|c} & \times & & \\ & \times & & o_1^0 \\ & \times & & \text{still!} \\ \hline & 0 & & 1 \end{array}\right]$$

because $A_2^1$ is a pure rotation

$$o_2^0 = A_2^0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \hline 1 \end{bmatrix} = \begin{bmatrix} a\cos q_1 - \cos q_1 \sin q_2 \\ a\sin q_1 - \sin q_1 \sin q_2 \\ -\cos q_2 \end{bmatrix}$$

Jacobian computation:

$$J_{v_1} = z_0 \times o_2^0 = \hat{k} \times o_2^0 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ o^0_{2x} & o^0_{2y} & o^0_{2z} \end{vmatrix}$$

$$J_{v_1} = \begin{bmatrix} s_1 s_2 - a s_1 \\ a c_1 - c_1 s_2 \\ 0 \end{bmatrix}$$

$$J_{v_2} = z_1 \times (o_2^0 - o_1^0) = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} a c_2 - c_1 s_2 - a c_1 \\ \cancel{a s_1} - s_1 s_2 - \cancel{a s_1} \\ -c_2 \end{bmatrix}$$

some computation required ...

$$J_{v_2} = \begin{bmatrix} -c_1 c_2 \\ -c_2 s_1 \\ s_2 \end{bmatrix}$$

so $$J_v = \begin{bmatrix} -s_1(a - s_2) & -c_1 c_2 \\ c_1 (a - s_2) & -c_2 s_1 \\ 0 & s_2 \end{bmatrix}$$

For $q_1 = 0$, $q_2 = -\pi/2$:

$$J_v = \begin{bmatrix} 0 & 0 \\ a+1 & 0 \\ 0 & -1 \end{bmatrix}$$

There is **NO** singularity because $J_v$ is <u>full-rank</u>

Manipulability: $$J_v^T J_v = \begin{bmatrix} (a+1)^2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sqrt{\det(J_v^T J_v)} = a+1$$

**b-** Angular velocity Jacobian:

$$J_{w_1} = z_0^0 = \hat{k} \; ; \quad J_{w_2} = z_1^0 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}$$

$$J_w = \begin{bmatrix} 0 & -\sin q_1 \\ 0 & +\cos q_1 \\ 1 & 0 \end{bmatrix}$$

the 2 columns are always linearly independent (sin $q_1$ and cos $q_1$ can't be zero at the same time)

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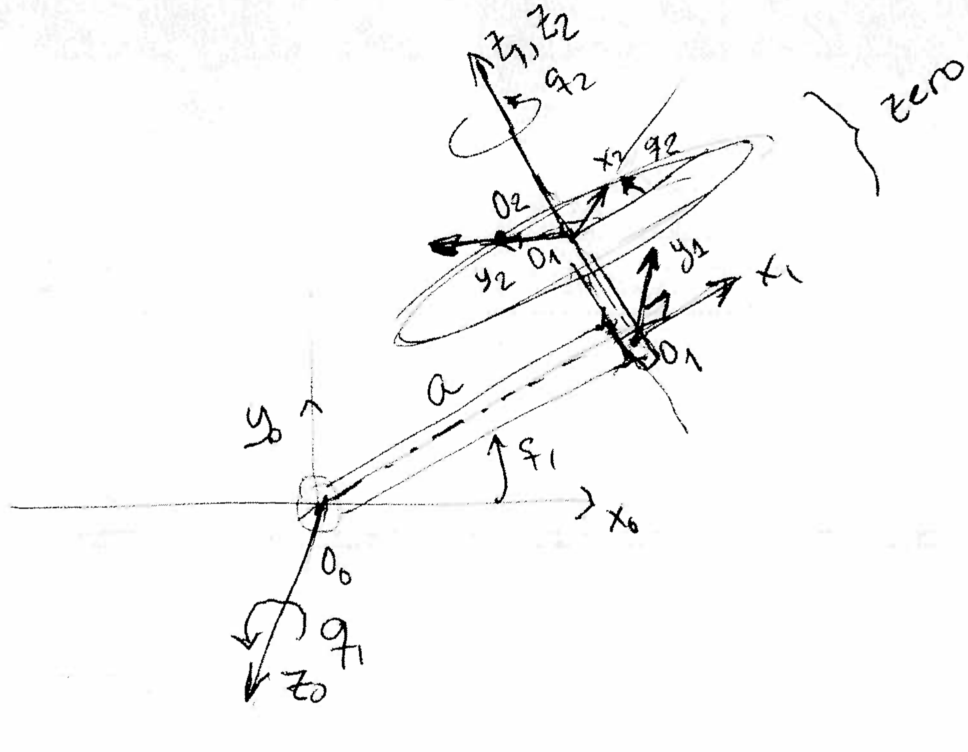
3b. Extended discussion (beyond the exam)



General singularities: If $s_2 = a \rightarrow$ singularity. ($|a| \leq 1$ needed)

Suppose $s_2 \neq a$:

$$J_v = \begin{bmatrix} -s_1(a-s_2) & -c_1 c_2 \\ c_1(a-s_2) & -c_2 s_1 \\ 0 & s_2 \end{bmatrix}$$

For the Jacobian to have rank less than 2

we need $s_2 = 0 \rightarrow c_2 = \pm 1$

so $J_v = \begin{bmatrix} -s_1 a & \pm c_1 \\ c_1 a & \pm s_1 \\ 0 & 0 \end{bmatrix}$

For the 2 columns to be dependent, we would also need $\begin{cases} -s_1 a = \pm c_1 \gamma \\ c_1 a = \pm s_1 \gamma \end{cases}$ for some scalar $\gamma$

- If $a = 0$ this is impossible, because $c_1$ and $s_1$ cannot be both zero.
- If $a \neq 0$: if $\gamma = 0$, singularity is impossible (same reason)

  If $\gamma \neq 0$: $\gamma = \pm \dfrac{s_1 a}{c_1}$, $\gamma = \pm \dfrac{c_1 a}{s_1}$ $\longrightarrow \dfrac{s_1^2}{c_1^2} = \pm 1$

of which only $\dfrac{s_1^2}{c_1^2} = 1$ is possible, so we would need $s_1^2 = c_1^2$

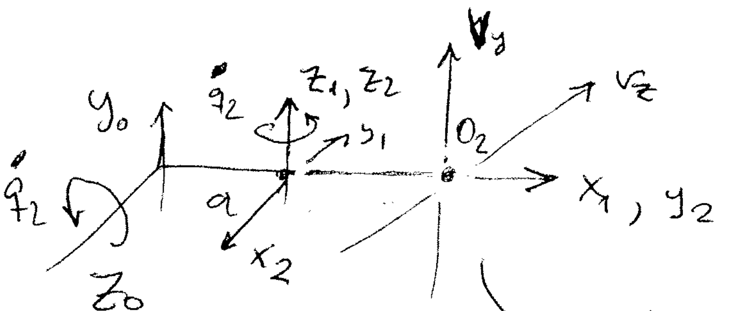
But the combination of signs for $s_1$ and $c_1$ in the columns of $J_v$ (for $s_2 = 0$) make this impossible.

$\rightarrow$ There are no linear velocity singularities, $\text{rank}(J_v) = 2$ for any $q_1, q_2$, unless $|a| \leq 1$, $\sin q_2 = a$

This means that the dimension of the velocity space for any configuration is always 2. (for $O_2$

Sometimes the velocity has all 3 $(x, y, z)$ components different than zero, sometimes not.

For example, for $q_1 = 0$, $q_2 = -\pi/2$




at this configuration, the absolute velocity of $O_2$ has nonzero $v_y$ & $v_z$ components, but zero $v_x$ component.

This is confirmed by finding the general

$$\mu(q) = \sqrt{\det(J_v^T J_v)}$$

$$J_v^T J_v = \begin{bmatrix} (a-s_2)^2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mu(q) = |(a-s_2)|.$$

$\mu(q) = 0$ only if $|a| \leq 1$ and $\sin(q_2) = a$

4. The closed-loop transfer function is

$$T(s) = \frac{\frac{1}{s^2}(K_D s + K_P)}{1 + \frac{K_D s + K_P}{s^2}}$$

$$T(s) = \frac{K_D(s + K_P/K_D)}{s^2 + K_D(s + K_P/K_D)}$$

If $K_P/K_D$ is very small ($\sim 0$)

$$T(s) = \frac{K_D s}{s^2 + K_D s} = \frac{K_D}{s + K_D}$$

- $T(0) = 1$, so there will be no offset (sse) for a step input reference.
- The percent overshoot will be zero, because $T(s)$ is first-order
- The time constant is $1/K_D$, so the settling time will be $\frac{4}{K_D}$.