MCE693/793: Analysis and Control of Nonlinear Systems

Introduction
Nonlinear Behavior

Hanz Richter
Mechanical Engineering Department
Cleveland State University
Linear vs. Nonlinear and the Anna Karenina Principle

Happy families are all alike; every unhappy family is unhappy in its own way.

Tolstoi, L., in “Anna Karenina”

- We will see that all linear systems are alike: single equilibrium, global stability, uniform scaling...
- There’s only one way to be “linear”, but many ways to be nonlinear..
Linear functions

A linear function (mapping) $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if it satisfies the superposition requirements:

1. Scaling (homogeneity): $f(\alpha x) = \alpha f(x)$ for all scalars $x$ and all $x \in \mathbb{R}^n$.
2. $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$.

Is the straight line $y = f(x) = x + 1$ a linear function?
Note: A function like $f(x, y) = x \sin(y)$ is said to be linear in $x$ (see $y$ as a constant and check linearity).
Linear dynamic systems

In this course we use “dynamic system” to refer to a system of differential equations with the general representation

\[ \dot{x} \triangleq \frac{dx}{dt} = f(x, u, t) \]

where \( x \) and \( u \) take values in \( \mathbb{R}^n \) and \( \mathbb{R}^m \). Thus, \( f \) maps \( \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \) into \( \mathbb{R}^n \).

A linear dynamic system corresponds to \( f \) linear in \( x, u \) and \( t \). Linearity may fail relative to some variables:

\[ f(x, u) = a(x) + b(x)u \]

with \( a(x) \) and \( b(x) \) nonlinear. Here linearity in control holds, it’s an affine (in control) system.
The above scaling property holds true for zero initial conditions. We analyze linearity of the system vs. linearity of the solution for the linear system in state space form:

\[ \dot{x} = Ax + Bu \quad y =Cx + Du \]

If the initial condition is not zero, do we still have uniform scaling?
Linear time-invariant (LTI) systems

Mathematical definition: Consider a system $y = Gx$ and suppose an input $x(t)$ produces and output $y(t)$. Then $G$ is time-invariant if the input $x(t + \tau)$ produces $y(t + \tau)$ regardless of $\tau$.
An absence of an explicit time dependence in $G$ guarantees time-invariance.

Representation: Every linear time-invariant system must be representable in the state-space form

$$\dot{x} = Ax + Bu \quad y = Cx + Du$$

The representation is not unique. For $G(s) = 1/s$, how could we find more than one representation?
Linear systems: equilibrium

For

\[ \dot{x} = Ax \]

deed are 2 possibilities:

1. If \( A \) is nonsingular, 0 is the only equilibrium point
2. If \( A \) is singular, there is an infinite number of equilibrium points (\( \text{null}(A) \)).

Then a linear system can’t have a finite number of nonzero equilibria (isolated points).
Linear systems: stability

For

\[ \dot{x} = Ax \]

Lyapunov stability (mathematical definition later): Trajectories remain confined in a set surrounding 0.

This is completely determined by the eigenvalues of \( A \). Lyapunov stability holds if \( \text{Re}(\lambda) \leq 0 \) for each \( \lambda \) such that \( |\lambda I - A| = 0 \).

For asymptotic stability (trajectories converge to zero): We require \( \text{Re}(\lambda) < 0 \).

Note that stability properties are immediately global (valid for all initial conditions) in linear systems.
Linear systems: forced response

Sinusoidal response: An asymptotically stable linear system with sinusoidal input produces a sinusoidal output component of the same frequency.

Each input frequency produces a single output amplitude and phase, regardless of previous input history. We say that linear systems have a unique frequency response.

\[ u(t) = u_0 \sin(\omega t) \quad y(t) = y_0 \sin(\omega t + \phi) \]

Also, there’s superposition (excludes initial condition response):

\[ G(u_1(t) + u_2(t)) = G(u_1(t)) + G(u_2(t)) \quad \text{and} \quad G(\alpha u(t)) = \alpha G(u(t)). \]
Linear systems: other features

- **Gaussian inputs**: A Gaussian noise is a signal whose values at any instant of time are drawn from a normal probability distribution (with constant mean and variance). A linear system produces Gaussian outputs in response to Gaussian inputs.

- **Oscillation scaling**: If an initial condition results in an oscillatory (periodic) response, then all nonzero initial conditions will result in an oscillation.

- Conversely, if an initial condition results in a non-oscillatory response, no other initial condition can bring about an oscillation.
Example from a piezoelectric tube actuator for nano-positioning:

Nonlinear systems: multiple and isolated equilibria

A nonlinear system may have an equilibrium set which is

- Empty: \( \dot{x} = 1 \)
- Infinite and dense: \( \dot{x} = \text{round}(x) \)
- Infinite and non-dense (isolated points): \( \dot{x} = \sin(x) \)
- Finite: \( \dot{x} = x^2 - 1 \)

Give a second example for each of the above systems, and for a nonlinear system having zero+dense set as equilibria (as in a singular linear system).
Nonlinear systems: stability

Stability in nonlinear systems is associated with equilibrium points, because confinement, convergence or divergence of trajectories depends on the point considered.

- Can stability be global for any given equilibrium point if there are multiple equilibria?
- Even if there’s a single stable equilibrium, stability doesn’t have to hold globally (sharp contrast with linear case)

Nonlinear systems can generate free oscillations which:

- Not always include the initial condition (contrast with orbits from linear harmonic oscillators)
- Are independent of the initial condition (starting from different points leads to the same steady oscillation). This doesn’t mean that all initial conditions produce the same orbit!

Discussion: Undamped harmonic oscillator (linear) vs. true pendulum.
True pendulum vs. harmonic oscillator

True pendulum:

\[ \ddot{x} + \frac{g}{l} \sin x = 0 \]

Harmonic oscillator:

\[ \ddot{x} + \frac{g}{l} x = 0 \]

- Both models generate steady orbits containing the initial condition. A given orbit cannot be reached from initial conditions other than points in the orbit (these are not limit cycles!)

- Problem: What are the constant energy loci in the phase plane for the linear oscillator and nonlinear pendulum for all possible initial conditions?
Nonlinear systems: forced response

In nonlinear systems, the scale of the input changes the scale of the output and it may produce qualitative changes also.

For instance, oscillations may occur only when the forcing input has a sufficiently large amplitude.

Easy example: mass-spring with stiction. If a sinusoidal force with small amplitude is applied, the response is zero. If the force is increased, an oscillation shows up.

We construct a simple model and a simulation in class.
Nonlinear systems: self-sustained oscillations

A self-sustained oscillation requires a forcing input, however, the input does not need to have the same frequency content as the oscillation.

In physical systems, a forcing input (could be constant) provides the energy necessary to overcome system damping and make the oscillation possible.

Example: Flutter. Aeroelastic oscillation occurs for a range of constant airspeeds.

https://www.youtube.com/watch?v=qpJBvQXQC2M
https://www.youtube.com/watch?v=OhwLojNerMU
Nonlinear systems: limit cycles

A limit cycle is a periodic oscillation (showing up in the state space as a closed orbit) with one of the following 2 properties:

1. There exist initial conditions outside the periodic oscillation resulting in trajectories converging to it.

2. There are trajectories originating in an arbitrarily small neighborhood of the periodic oscillation which escape to infinity or converge to another limit cycle or stationary point.

If there is a neighborhood of initial conditions resulting in converging trajectories, the L.C. is stable.
If all neighborhoods of initial conditions (excluding the LC itself) result in diverging trajectories, the L.C. is unstable.
Nonlinear systems: limit cycles

A limit cycle can be produced by forced or unforced nonlinear systems.

Classical unforced example: van der Pol equation:

\[ \ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \]
Nonlinear systems: frequency response

Suppose a sinusoidal input is applied to a nonlinear system, resulting in a periodic response.

- Unlike linear systems, the output doesn’t necessarily have the same frequency as the input.
- More than one frequency may be present in the output. In some cases no output frequency component coincides with the input frequency. Easy example: squaring nonlinearity.
- There can be *subharmonics*: components with frequencies that are integer submultiples of the driving frequency. See Hayashi, Ch., J. Appl. Phys. (1953) [http://dx.doi.org/10.1063/1.1721322](http://dx.doi.org/10.1063/1.1721322)
- Worse: Frequency response may be non-unique: Jump resonance.
Nonlinear systems: jump resonance

Imagine an experiment in which a sequence of sinusoidal inputs with different frequencies but the same amplitudes is applied to a system. Suppose a steady periodic solution is obtained in each case, where the fundamental output component has the same frequency as the input.

For each (discrete) frequency, the fundamental output amplitude is recorded. In a linear system, amplitudes are a function of frequency, and the same function is observed regardless of the order of application of the input frequencies.

A nonlinear system could be “sensitive” to the input application history (through the state of the system at the time of input frequency changes). A given frequency does not uniquely define an output amplitude.

Mathematical analysis with *describing functions* (later in course) can be used to predict the existence of up to three steady amplitude solutions for the same frequency. However, the appearance of a specific steady amplitude is a very complex dynamic process, governed by the initial conditions in the system and the input frequency application sequence, as well as the input amplitude.

Nonlinear systems: stochastic response

- A Gaussian input applied to a nonlinear system results in a non-Gaussian output in general.

- Dual-mode operation: The variance of the output may present "jumps", depending on how the variance of the input is changed. This is linked to jump resonance.
Nonlinear systems: chaos and bifurcations

Chaotic behavior in a system is characterized by:

■ Sensitivity to initial conditions: a small change in initial conditions may result in a major change in future evolution (butterfly effect).

■ Non-periodicity, pseudo-randomness: Although the system is defined deterministically (absence of randomness), its response appears “unpredictable” and random-like (three-body problem).

■ Bifurcations: a smooth change in a system parameter can bring about an abrupt *qualitative* change in system properties, for instance the nature of an equilibrium point (from stable to unstable).

https://www.youtube.com/watch?v=f-acXTWEXZI
Even a simple system like the free double pendulum is capable of displaying the above characteristics.