MCE693/793: Analysis and Control of Nonlinear Systems

Input-Output and Input-State Linearization

Zero Dynamics of Nonlinear Systems

Hanz Richter
Mechanical Engineering Department
Cleveland State University
Control by Exact Linearization

Input to state linearization, also called exact linearization, if possible, provides a simple way to specify control actions.

For instance, a 3rd-order feedback linearizable system \( \dot{x} = f(x) + g(x)u \) can be written as:

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_f &= v
\end{align*}
\]

where \( u = u(z, v) \) is a suitable input transformation and \( v \) is a new control.

Then control design becomes trivial using \( v \) as the control. The state and control transformations are used to express the control law in original coordinates.
Complete Integrability

Refer to Slotine and Li, Sect. 6.2 for the motivation of the definition below.

Suppose the set \( \{f_i(x)\}, i = 1, 2...m \) is linearly independent in some subset \( U \subset \mathbb{R}^n \). The set is said to be completely integrable in \( U \) if we can find \( n - m \) scalar functions \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) satisfying:

\[
\nabla h_i f_j = 0
\]

for \( 1 \leq i \leq n - m, 1 \leq j \leq m \), with the set of \( \{\nabla h_i(x)\} \) linearly independent in \( U \).

Under this definition, we have \( m(n - m) \) partial differential equations. If there’s a solution, the set of vector fields is completely integrable.

Exercise: Determine if the following set is completely integrable (can you find an subset of \( \mathbb{R}^n \) where the vector fields are linearly independent?)

\[
f(x) = [\sin(x_1) \ 0 \ \cos(x_2)]^T, \quad g(x) = [\cos(x_3) \ x_2 \ \sin(x_2)]^T
\]
Involutivity and Frobenius Theorem

Suppose that for any pair of vector fields $f_i, f_j$ from a linearly independent set \{f_1, f_2, \ldots, f_m\}, the Lie bracket $[f_i, f_j]$ can always be expressed as a linear combination of the vector fields in the set (with possibly state-dependent coefficients). Then the set is involutive.

The test for involutivity is just a pointwise rank test: for any $i, j$, $[f_i(x), f_j(x)]$ must already be in the span of \{f_1(x), f_2(x), \ldots, f_m(x)\} for all $x$ in the subset where involutivity is to be verified.

Example: Use a numerical or analytical procedure to test involutivity of the set \{f, g\} of the previous example in the same region where these fields are linearly independent.

Frobenius Theorem: The linearly independent set \{f_1, f_2, \ldots, f_m\} is completely integrable if and only if it is involutive.
Consider the system
\[ \dot{x} = f(x) + g(x)u \]
where \( f \) and \( g \) are smooth vector fields. The system is \textit{input-to-state, or exactly linearizable} if there is a diffeomorphic state transformation \( \Phi : \Omega \mapsto \mathbb{R}^n \) with \( z = \Phi(x) \) and an input transformation \( u = \alpha(x) + \beta(x)v \) in some region \( \Omega \subset \mathbb{R}^n \) such that the new state \( z \) and the new input \( v \) are related by the linear, time-invariant state equation below:
\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_n &= v
\end{align*}
\]

Note that there’s no output defined, and if this ambitious transformation is possible, there are no internal dynamics left. Among mechanical systems, robot dynamics are an important case of input-state linearizable systems with many inputs.
Necessary and Sufficient Condition for Exact Linearizability

(Theorem 6.2 in Slotine and Li): The single-input, $n$-state nonlinear system
\[ \dot{x} = f(x) + g(x)u \]
is exactly linearizable iff the following conditions hold in some region $\Omega \subset \mathbb{R}^n$:

1. The vector fields \( \{g, \text{ad}_f g, ... \text{ad}_{f^{n-1}} g\} \) are linearly independent in $\Omega$.

2. The set \( \{g, \text{ad}_f g, ... \text{ad}_{f^{n-2}} g\} \) is involutive in $\Omega$.

What happens with these conditions for linear systems ($f(x) = Ax$ and $g(x) = B$)?

The proof of this Theorem includes a method to construct the diffeomorphic state transformation and the input transformation. Suppose the system is exactly linearizable and let $z(x) = [z_1(x) z_2(x) ... z_n(x)]^T$. Find $z_1$ first, to satisfy

\[ \nabla z_1 \text{ ad}_{f^i} g = 0, \quad i = 1, 2..n - 2 \]
\[ \nabla z_1 \text{ ad}_{f^{n-1}} g \neq 0 \]
Recipe for Exact Linearizability...

The state transformation is

\[ z(x) = [z_1 \ L_f z_1 \ ... \ L_{f_n - 1} z_1]^T \]

while the input transformation uses

\[
\alpha(x) = -\frac{L_f^n z_1}{L_g L_{f_{n-1}} z_1} \\
\beta(x) = \frac{1}{L_g L_{f_{n-1}} z_1}
\]

Examples: We solve Problems 6.3 and 6.7 from Slotine and Li.
Input-Output Linearization

Consider the affine system

\[
\dot{x} = f(x) + g(x)u, \quad y = h(x)
\]

We want to define a new input \( v \) such that \( y \) is related to it by linear dynamics in some subset \( \Omega \) of the state space. We do this to select a control law \( v \) with linear methods. Then we transform \( v \) to the original control \( u \).

Differentiate the output repeatedly, until \( u \) appears with a nonzero “coefficient”:

\[
\begin{align*}
\dot{y} &= L_fh + (L_gh)u = L_fh \\
\ddot{y} &= L_{f2}h + L_g(L_fh)u = L_{f2}h \\
&\vdots \quad \vdots \\
\frac{d^r y}{dt^r} &= L_{fr}h + L_g(L_{fr-1}h)u
\end{align*}
\]

where the process stops for the smallest \( r \) such that \( L_g(L_{fr-1}h) \) doesn’t vanish in \( \Omega \).
Undefined Relative Degree

It is possible that $L_g(L_{fr-1} h)$ is zero at a point $x_0 \in \Omega$ and nonzero at every other point of this set.

This makes the relative degree undefined. This is important when $x_0$ is the desired point of regulation or if system trajectories cannot be guaranteed to approach $x_0$.

An I/O linearization design conducted by ignoring this fact can lead to erroneous results (see example later).
Input-Output Linearization...

From
\[
\frac{dy^r}{dt^r} = L_fh + L_g(L_{fr-1}h)u
\]

we can define a new control \( v \) as the right-hand side of the above expression and find \( u \):
\[
u = \frac{1}{L_g(L_{fr-1}h)}(v - L_fh)
\]

This control law reduces the system to
\[
\frac{dy^r}{dt^r} = v
\]

and the number \( r \) is called the \textit{relative degree}. For linear systems, \( r \) is the difference between the number of poles and zeros of the associated transfer function.

- When \( r = n \), we actually achieve input-state linearization and there are no internal dynamics.
- When \( r < n \), there are internal dynamics not expressed by the linearized input-output relationship.
Zeros and Internal Dynamics: Linear Systems

For a single-output, single input linear system

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

the zeros of the transfer function correspond to the poles of the internal dynamics (if any) when the output is constrained to zero.

As an example, take

\[
G(s) = \frac{s + a}{s^2 + s + 1}
\]

find a state space realization and restrict the output to zero, to find the reduced-order dynamics. Compare them with the numerator of \(G(s)\).

Internal dynamics are also referred to zero dynamics.

Zeros in multivariable systems require a more involved definition.
Internal Dynamics Stability: Minimum/Non-Minimum Phase

When the zero dynamics of a linear system are stable (zeros in LHP), we say the system is “minimum phase”.

This because of the fact that two systems could coincide in magnitudes at all frequencies, but have different phases:

\[ G(s) = \frac{1 - s}{1 + 2s} \]

vs.

\[ G(s) = \frac{1 + s}{1 + 2s} \]

The system with the zero in the RHP has a larger phase variation: it’s a non-minimum phase system.

Internal stability is crucial for systems with relative degree less than \( n \) with controllers that attempt to regulate the output to zero (sliding mode control).
Nonlinear Zero Dynamics

The zero dynamics of the nonlinear system

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]

are the reduced-order dynamics resulting from imposing the constraint \( y = 0 \) for all times.

When performing design on the basis of input/output linearization (also known as feedback or exact linearization), we must check the stability of any residual internal dynamics.

Note: Robotic systems

\[
M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u
\]

have relative degree \( n \) and are trivially feedback linearizable.
Example: Successful I/O linearization design

Consider

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \sin(x_1) + (1 + x_2^2)u \\
y &= x_1
\end{align*}
\]

We verify that the control transformation

\[
u = \frac{v - \sin(y)}{1 + \dot{y}^2}
\]

puts the system in double-integrator form. The virtual control can be chosen for any objective (tracking, regulation, etc.). For regulation we may choose:

\[
v = -y - \dot{y}
\]

Here \( r = n \), so there are no residual internal dynamics. If \( y(t) \) is set to a constant \( c \) for all times, the system is described by:

\[
\dot{x}_2 = \dot{x}_1 = 0; \quad x_1 = c
\]
Example: Undefined relative degree

Consider

\[
\begin{align*}
\dot{x}_1 &= x_1 x_2 \\
\dot{x}_2 &= x_2 + u \\
y &= x_1
\end{align*}
\]

Two differentiations of the output are required for the control input to appear:

\[
\ddot{y} = x_1 (x_2^2 + x_2) + x_1 u
\]

However, the coefficient of \(u\) is zero at the isolated point \(x_1 = 0\). The relative degree is not 2 in any neighborhood of the origin.
Example...

The control transformation

\[ u = \frac{v - x_1 x_2^2}{x_1} - x_2 \]

puts the system in double-integrator form. For regulation to zero we could incorrectly choose:

\[ v = -x_1 - \dot{x}_1 \]

We see that the control becomes singular as the origin is approached. Also, to maintain \( y(t) = 0 \) at all times, we need

\[ u = -x_2 - x_2^2 \]

The zero dynamics are unstable (a separate issue):

\[ \dot{x}_2 = -x_2^2 \]

We needed two differentiations of \( y \) for \( u \) to appear. However, because the relative degree is not 2, there are internal dynamics.
Example

Consider the two-mass system with one nonlinear spring shown below:

We analyze I/O linearizability and zero dynamics relative to the output
$y = x_1 - x_2$ (distance between masses).

Can we stabilize $y$ with I/O linearization techniques?

Is the system I/S linearizable? Can we use I/S linearization to regulate $y$ to a desired value?